WHAT IS A SURD?

Mathematics is one subject where students can learn about the importance of definitions. Definitions should be unambiguous and accurate. In this article we consider the definition of a surd as presented by numerous sources.

In 1886, the Scottish mathematician George Chrystal published a now classic work on algebra. Chrystal used the term 'commensurable' rather than 'rational', and 'incommensurable' rather than 'irrational'. He said that a commensurable number is 'not a perfect *n*th power' if it is not the *n*th power of another commensurable number. Chrystal (1964, Part 1, p. 203) defined a surd number as follows.

> Let p be any *commensurable number* ... If therefore p be not a perfect nth power, $\sqrt[n]{p}$ is incommensurable. For distinction's sake $\sqrt[n]{p}$ is then called a *surd number*. In other words, we define a *surd number as the incommensurable root of a commensurable number*.

Notice that Chrystal used the term 'surd number' rather than 'surd'. The wording would be modernised if 'commensurable' were replaced by 'rational' and 'incommensurable' by 'irrational'.

Let us now turn to the Australian Curriculum. In the mathematics glossary associated with the Australian Curriculum v.8.1, F-10, a surd is defined as follows.

> A surd is a numerical expression involving one or more irrational roots of numbers. Examples of surds include $\sqrt{2}$, $\sqrt[3]{5}$ and $4\sqrt{3}+7\sqrt[3]{6}$.

Furthermore, the glossary defined an expression as follows.

> … two or more numbers or variables connected by operations. For example, 17 – 9, 8 × (2 + 3), $2a + 3b$ are all expressions. Expressions do not include an equal sign.

It is interesting that the Australian Curriculum defines a surd as a particular type of 'numerical expression' rather than a number. According to this definition, it appears that $\sqrt{2}-\sqrt{2}$, $5+\sqrt{2}-\sqrt{2}$, $\sqrt{\pi}$, and $\left(\sqrt{2}\right)^2$ would all be surds because these are numerical expressions that involve one or more irrational roots of numbers.

AMSI (2011) provided the following definition.

> If a is a rational number, and n is a positive integer, any irrational number of the form $\sqrt[n]{a}$ will be referred to as a surd. A real number such as $2\sqrt{3}$ will be loosely referred to as a surd, since it can be expressed as $\sqrt{12}$. For the most part, we will only consider quadratic surds, $\sqrt{a}$, that involve square roots. We will also say that $\sqrt{2}+\sqrt{3}$ is a surd, although technically we should say that it is the sum of two surds.

The first sentence suggests that a surd is a number, but the second sentence states that $2\sqrt{3}$ should be 'loosely referred to as a surd' because it is equal to the number $\sqrt{12}$. Does this mean that a surd is an expression rather than a number? The final sentence detracts from the clarity of the first.

Why should we say that $\sqrt{2}+\sqrt{3}$ is a surd? Would we say that $\sqrt{2}-\sqrt{2}$ is a surd?

In a contemporary text-book, Garner et al. (2012a. p. 26) stated

> A surd is an irrational number that is the root of a rational number, such as $\sqrt{5}$.

This makes it clear that a surd is a number rather than an expression, however, the definition does not make it clear that the root in question can be something other than a square root.

In another text-book, Garner et al. (2012b, p. 319) stated

> A surd is an irrational number that can only be expressed exactly using the square root sign.

Again a surd is regarded as a number. However, the definition allows only square roots. Further confusion arises from the word 'only'. $\sqrt{2}$ can also be expressed exactly as $\sec\left(\frac{\pi}{4}\right)$ which does not involve the square root sign.

Coffey et al. (2013, p. 676) defined a surd as 'a number (irrational) that can only be expressed exactly using the radical sign.' The reservations expressed in the preceding paragraphs apply again.

In their dictionary of mathematics, De Klerk and Marasco (2013) defined a surd as

> … an irrational number that can only be expressed exactly by using the root symbol, $\sqrt{\ }$ (also known as the radical symbol).

They gave, as examples, $\sqrt{29}$, $\sqrt[3]{10}$, $2\sqrt{5}$, an expression such as $\sqrt{6}+\sqrt{3}$. Again, many of the comments made above apply to this definition.

Evans et al. (2016, p. 51) gave the following definition.

> A quadratic surd is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

In this definition the irrationality of a surd follows from the definition rather than being built in to the definition. The authors continue

> In general, a surd of order n is a number of the form $\sqrt[n]{a}$ where a is a rational number which is not a perfect n^{th} power.

The last clause should read, 'which is not a perfect n^{th} power of another rational number'.

A delightful, almost poetic, definition was presented by Howard Eves (1990, p. 241).

> A numerical radical in which the radicand is rational but the radical itself irrational is called a surd. A surd is called quadratic, cubic, and so on, according as its index is 2, 3, and so on.

Is $-\sqrt{2}$ a surd? None of the references mentioned so far say anything specific about this question.

The classic work by G.H. Hardy (1952, p. 20) which was first published in 1908 is informative on this matter. Hardy wrote, in terms of quadratic surds (those that involve only square roots)

> A number of the form $\pm\sqrt{a}$ where a is a positive rational number which is not the square of another rational number, is called a pure quadratic surd. A number of the form $a \pm \sqrt{b}$, where a is rational, and $\sqrt{b}$ is a pure quadratic surd, is sometimes called a mixed quadratic surd.

Hardy seems to regard a surd as a number expressed in a particular form.

Smith (1958, p. 252) states

> As to what constitutes a surd, however, there has never been a general agreement.

This was originally published in 1925 for teachers and students. Things have not changed in the last 90 years. There is still considerable variation in published statements on the meaning of a surd. It is not even clear whether a surd is a number or an expression. Polster and Ross (2010) remark, 'There is broad disagreement on exactly which irrational numbers are surds.' This unwanted variation is not helpful to the many students and teachers who rely heavily on the written word.

OUR DEFINITION

We suggest the following definition.

A surd number is a number that can be expressed in the form $\pm\sqrt[n]{a}$ where $n \geq 2$ is an integer and a is a positive rational number which is not the nth power of a rational number. The expression $\pm\sqrt[n]{a}$ will be referred to as a surd expression.

Should the term 'surd' be used at all in the school syllabus? Surds are used in schools primarily for teaching students about the algebraic manipulation of roots, and the irrationality of a surd almost never comes into consideration. Furthermore, the term 'surd' is not used in applications of mathematics, or in mathematics encountered after secondary school. We recommend that the term 'surd' be deleted from the Australian Curriculum.

AUTHORS



Christopher Lenard, Terence Mills and Tina Fitzpatrick are from La Trobe University. Christopher and Terence are in the Department of Mathematics and Statistics, and Tina is in the Department of Education.

Email: C.Lenard@latrobe.edu.au, T.Mills@latrobe.edu.au, T.Fitzpatrick@latrobe.edu.au

REFERENCES

(2016, May). Australian Curriculum v. 8.1, F-10 Curriculum. Retrieved from www.australiancurriculum.edu.au/mathematics/glossary

(2011, June). Australian Mathematical Sciences Institute (AMSI) Surds. Retrieved from http://amsi.org.au/teacher_modules/Surds.html#Content

Chrystal, G. (1964). *Algebra: An Elementary Text-Book for the Higher Classes of Secondary Schools and for Colleges.* (7th ed.) New York: Chelsea Publishing Company

Coffey, D. et al. (2013). Pearson Mathematics SB 9. Melbourne: Pearson Australia

Evans, M., Wallace, D., Lipson, K., & Treeby, D. (2016). *Specialist Mathematics, Units 1 & 2*. Port Melbourne: Cambridge University Press.

Eves, H. (1990). *An Introduction to the History of Mathematics*, (6th ed.) Pacific Grove, CA: Brooks-Cole-Thomson

Garner S., Brodie R., Corcoran S., & Swift S. (2012). *Nelson Think Maths for the Australian Curriculum, 10+10A*. South Melbourne: Cengage Learning

Garner S., Brodie R., Corcoran S., & Swift S. (2012). *Nelson Think Maths for the Australian Curriculum, 9*. South Melbourne: Cengage Learning

Hardy, G.H. (1952). *A Course of Pure Mathematics*. (10th ed.) Cambridge: Cambridge University Press

De Klerk, J., & Marasco, A. (2013). *Pearson Illustrated Maths Dictionary*. (5th ed.) Melbourne: Pearson Australia

Polster, B., & Ross, M. (2010, May 24). Theatre of the Surd. *The Age.*

Smith, D.E. (1958). *History of Mathematics, Vol. II*. New York: Dover