COMPLEX NUMBERS

Content: Construction of the complex number system; review of complex numbers; Euler's formula; trigonometric identities and their application; the complex exponential; differentiation and integration via the complex exponential.
(Complex trigonometric, hyperbolic and inverse trigonometric functions).

Construction of a number system in which there is a number *i* with the property that $i^2 = -1$:

For the construction a temporary notation is used which will later be abandoned.

A complex number z is an ordered pair $[x, y]$ of real numbers with the following definitions of addition and multiplication:

$$[x, y]+[x', y'] = [x+x', y+y'], \qquad [x, y]\times[x', y'] = [xx'-yy', xy'+yx'].$$

Theorem: The operations of addition and multiplication defined above satisfy the commutative, associative and distributive laws.

The *additive identity* is $[0,0]$ and the *multiplicative identity* is $[1,0]$.

The *additive inverse* of $[x, y]$ is $[-x,-y]$ and can be used to define subtraction: $[x, y]-[x', y'] = [x, y]+[-x',-y']$.

Theorem: Complex numbers are closed under the operations of addition and subtraction defined above.

The complex number $[x,0]$ behaves exactly like the real number x under addition and multiplication:

$$[x,0]+[x',0] = [x+x',0], \qquad [x,0]\times[x',0] = [xx',0].$$

Henceforth the complex number $[x,0]$ is identified with the real number x so that by construction $R \subset C$.

Note that $[y,0]\times[0,1] = [0, y]$, so that

$$[x, y] = [x,0]+[0, y] \qquad = [x,0]+[y,0]\times[0,1] \qquad = x + y[0,1]$$

using the identification $[x,0] \leftrightarrow x$ and $[y,0] \leftrightarrow y$.

Finally note that

$$[0,1]\times[0,1] = [-1,0]$$

and $[-1,0]$ is identified as -1. The imaginary unit i is defined by

$$i = [0,1].$$

Thus by construction:

(a) Every complex number $z = [x, y]$ can be written in the form $x+iy$ where x and y are real numbers.

(b) $i^2 = -1$.

Note on square roots: $[0,1]$ and $[0,-1]$ are both square roots of -1. The $[0,1]$ root is defined to be i.

The temporary notation $[x, y]$ is now abandoned.

Provided that $i^2 = -1$ is recalled, all the usual laws of addition, subtraction and multiplication apply to complex numbers.

The *multiplicative inverse* and division: For any non-zero $z = x+iy$ there is a unique number z^{-1} such that $zz^{-1} = z^{-1}z = 1$.

Let $z^{-1} = [u,v]$ be the multiplicative inverse of $z = [x, y]$.

Then $[x, y]\times[u,v] = [1,0] \quad \Rightarrow [xu-yv, yu+xv] = [1,0] \quad \Rightarrow u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$.

Thus $z^{-1} = \left[\dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2}\right]$.

The multiplicative inverse can be used to define division by a non-zero complex number: $\dfrac{z_1}{z_2} = z_1 \times z_2^{-1}$.

Calculations such as $\dfrac{2-3i}{1+2i} = \dfrac{2-3i}{1+2i}\times\dfrac{1-2i}{1-2i} = \dfrac{(2-3i)(1-2i)}{1^2+2^2}$ are now justified.

Review: • Definition: $i^2 = -1$; real and imaginary parts; equality; the complex plane and argand diagrams; complex conjugate; magnitude (modulus); argument and principal argument.

• Cartesian form: Representation; addition and subtraction; multiplication; division.

• Polar form: Representation; multiplication and division; De Moivre's Theorem; geometric meaning of multiplication by i; calculation of powers and roots.

A note on square roots: When $a \in R^+$, by convention $\sqrt{a}$ is the positive number r such that $r^2 = a$ *eg*. $\sqrt{4} = 2$. But when $a \in C$ the two square roots cannot be distinguished in this way because (**theorem:**) the set of complex numbers is not an ordered field. To avoid the ambiguity created by the above convention, $\sqrt{z}$ is defined to be single-valued (the *principal square root*): If $z = re^{i\theta}$ and $-\pi < \theta \le \pi$, then $\sqrt{z} = \sqrt{r}e^{i\theta/2}$. In particular $\sqrt{-1} = i$ and $\sqrt{-a} = i\sqrt{a}$ when $a \in R^+$.

Under this definition $\sqrt{ab} \ne \sqrt{a}\sqrt{b}$ if both $a, b < 0$ and $\sqrt{\frac{a}{b}} \ne \frac{\sqrt{a}}{\sqrt{b}}$ if one of $a, b < 0$ (contradictions arise).

• Polynomials: Fundamental Theorem of Algebra; Conjugate Root Theorem; factorisation and solutions over C of polynomials and polynomial equations (real coefficients and non-real coefficients).