MATHEMATICAL METHODS 2019 VCAA EXAMINATION 2

Question 3 part e. - SUGGESTED SOLUTION

The solution below explains where the transformations stated in the solution given by the VCAA Examination Report (and the itute solutions) for **Question 3 part e** come from.

Note: The values for *a*, *b* and *c* resulting from these transformations is not a complete set of values: There are an infinite number of values arising from an infinite number of families of solutions. The VCAA Examination Report gives two such families of solutions and states without further explanation that "There are other solutions."

From the graph given in the preamble it is seen that the area between the graph of *f* and the horizontal axis for $t \in [0, 6]$ is given by

$$\int_0^4 f(t)\,dt - \int_4^6 f(t)\,dt = \int_6^8 f(t)\,dt - \int_8^{12} f(t)\,dt .$$

Option 1: Consider $\int_6^8 f(t)\,dt - \int_8^{12} f(t)\,dt$.

If the graph of *f* is translated horizontally then the area stays the same but the integral terminals change. In particular, translating *f* to the left by 6 units gives

$$\int_6^8 f(t)\,dt - \int_8^{12} f(t)\,dt = \int_0^2 f(t+6)\,dt - \int_2^6 f(t+6)\,dt .$$

Furthermore, since the period of *f* is 12 (see **part a.**), further translations by integer multiples of 12 can be made without changing the integral terminals since $f(t+6) = f(t+6+12n)$ where $n \in Z$:

$$\int_0^2 f(t+6)\,dt - \int_2^6 f(t+6)\,dt = \int_0^2 f(t+6+12n)\,dt - \int_2^6 f(t+6+12n)\,dt \qquad \text{where } n \in Z .$$

Note that $\int_2^0 g(t)\,dt + \int_2^6 g(t)\,dt = -\int_0^2 g(t)\,dt + \int_2^6 g(t)\,dt$.

Compare $-\int_0^2 g(t)\,dt + \int_2^6 g(t)\,dt$ with $\int_0^2 f(t+6+12n)\,dt - \int_2^6 f(t+6+12n)\,dt$:

$$-g(t) = f(t+6+12n) \qquad \Rightarrow g(t) = -f(t+6+12n) .$$

Therefore $g(t)$ is the reflection of $f(t)$ in the *t*-axis followed by a horizontal translation to the left by $6+12n$ units (the order of these two transformations does not matter):

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -6-12n \\ 0 \end{bmatrix} \text{ where } n \in Z .$$

Option 2: Consider $\int_0^4 f(t)\,dt - \int_4^6 f(t)\,dt$.

If the graph of *f* is reflected in the *y*-axis then the area stays the same but the integral terminals change:

$$\int_0^4 f(t)\,dt - \int_4^6 f(t)\,dt = \int_{-4}^0 f(-t)\,dt - \int_{-6}^{-4} f(-t)\,dt .$$

If the graph of *f* is translated horizontally then the area stays the same but the integral terminals change. In particular, translating *f* to the right by 6 units gives

$$\int_2^6 f(-t)\,dt - \int_0^2 f(-t)\,dt = \int_2^6 f(-(t-6))\,dt + \int_2^0 f(-(t-6))\,dt = \int_2^0 f(-t+6)\,dt + \int_2^6 f(-t+6)\,dt .$$

Furthermore, since the period of *f* is 12, further translations to the right by integer multiples of 12 can be made without changing the integral terminals:

$$\int_2^6 f(-(t-6-12n))\,dt + \int_2^0 f(-(t-6-12n))\,dt = \int_2^0 f(-t+6+12n)\,dt + \int_2^6 f(-t+6+12n)\,dt \quad \text{where } n \in Z .$$

Compare $\int_2^0 g(t)\,dt + \int_2^6 g(t)\,dt$ with $\int_2^0 f(-t+6+12n)\,dt + \int_2^6 f(-t+6+12n)\,dt$:

$g(t) = f(-t+6+12n)$.

Therefore $g(t)$ is the reflection of $f(t)$ in the *y*-axis followed by a horizontal translation to the right by 6 units.

Therefore $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6+12n \\ 0 \end{bmatrix}$ where $n \in Z$.

Note: The values for *a*, *b* and *c* resulting from these transformations is not a complete set of values. There are an infinite number of values arising from an infinite number of families of solutions. The complete set of solutions is found as follows:

$$T\left(\begin{bmatrix} t \\ y \end{bmatrix}\right) = \begin{bmatrix} t' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} t \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \qquad \Rightarrow \begin{cases} x = \dfrac{t'-c}{a} \\ y = \dfrac{y'-d}{b} \end{cases}$$

$$\Rightarrow \frac{y'-d}{b} = f\left(\frac{t'-c}{a}\right) \qquad \Rightarrow y = g(t) = bf\left(\frac{t-c}{a}\right) + d .$$

Therefore:

$$\int_2^0 g(t)\ dt + \int_2^6 g(t)\ dt = \frac{15}{\pi} \quad \text{(from **part d**)}$$

where $g(t) = bf\left(\frac{t-c}{a}\right) + d$ and $f(t) = \sin\left(\frac{\pi}{3}t\right) + \sin\left(\frac{\pi}{6}t\right)$.

For example:

Some answers from 'families of solutions known to VCAA':

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In[1002]:= Clear[f]

In[1003]:= f[t_] := Sin[Pi / 3 * t] + Sin[Pi / 6 * t]

In[1006]:= Clear[g]

In[1007]:= g[t_, a_, b_, c_, d_] := b * f[(t - c) / a] + d

In[1008]:= Integrate[g[t, 1, -1, -6, 0], {t, 2, 0}] + Integrate[g[t, 1, -1, -6, 0], {t, 2, 6}]

Out[1008]= 15/π

In[1025]:= Integrate[g[t, 1, -1, -6 + 12, 0], {t, 2, 0}] + Integrate[g[t, 1, -1, -6 + 12, 0], {t, 2, 6}]

Out[1025]= 15/π

In[1009]:= Integrate[g[t, -1, 1, 6, 0], {t, 2, 0}] + Integrate[g[t, -1, 1, 6, 0], {t, 2, 6}]

Out[1009]= 15/π

In[1011]:= Integrate[g[t, -1, 1, 6 + 12, 0], {t, 2, 0}] + Integrate[g[t, -1, 1, 6 + 12, 0], {t, 2, 6}]

Out[1011]= 15/π

In[1029]:= Solve[Integrate[g[t, -1, 1, c, 0], {t, 2, 0}] + Integrate[g[t, -1, 1, c, 0], {t, 2, 6}] == 15 / Pi, c, Reals]

Out[1029]= {{c → 6 + 12 c1 if c1 ∈ ℤ}}

In[1030]:= Solve[Integrate[g[t, 1, -1, c, 0], {t, 2, 0}] + Integrate[g[t, 1, -1, c, 0], {t, 2, 6}] == 15 / Pi, c, Reals]

Out[1030]= {{c → 6 + 12 c1 if c1 ∈ ℤ}}
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Some answers from 'families of solutions **unknown to VCAA**':

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In[1032]:= Solve[{Integrate[g[t, -2, 1, c, 0], {t, 2, 0}] + Integrate[g[t, -2, 1, c, 0], {t, 2, 6}] == 15 / Pi, -20 < c < 20}, c, Reals] //
             N

Out[1032]= {{c → 7.51265}, {c → -16.4874}, {c → 9.64048}, {c → -14.3595}}

In[1034]:= Solve[{Integrate[g[t, -2, 3, c, 0], {t, 2, 0}] + Integrate[g[t, -2, 3, c, 0], {t, 2, 6}] == 15 / Pi, -20 < c < 20}, c, Reals] //
             N

Out[1034]= {{c → -6.2971}, {c → 17.7029}, {c → -3.18031}, {c → 5.87247}, {c → -18.1275}, {c → 11.6049}, {c → -12.3951}}
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