Question 4

Method 1:

$\cot(2x) = \sec(x)$ .... (1)

$\Rightarrow \tan(2x) = \cos(x) \qquad \Rightarrow \dfrac{\sin(2x)}{\cos(2x)} = \cos(x)$

$\Rightarrow \sin(2x) = \cos(x)\cos(2x)$

$\Rightarrow 2\sin(x)\cos(x) = \cos(x)\cos(2x)$

$\Rightarrow 2\sin(x)\cos(x) - \cos(x)\cos(2x) = 0$

$\Rightarrow \cos(x)\big(2\sin(x) - \cos(2x)\big) = 0$.

Case 1: $\cos(x) = 0 \qquad \Rightarrow x = \dfrac{\pi}{2} + n\pi,\ n \in Z$.

But $\sec(x) = \dfrac{1}{\cos(x)}$ and $\cot(2x) = \dfrac{\cos(2x)}{\sin(2x)}$ in equation (1) are not defined for $x = \dfrac{\pi}{2} + n\pi$ therefore $x = \dfrac{\pi}{2} + n\pi$ is rejected.

Case 2: $2\sin(x) - \cos(2x) = 0$.

Substitute $\cos(2x) = 1 - 2\sin^2(x)$:

$2\sin(x) - \big(1 - 2\sin^2(x)\big) = 0$

$\Rightarrow 2\sin^2(x) + 2\sin(x) - 1 = 0$. .... (2)

Equation (2) is a quadratic in $\sin(x)$.

Use the quadratic formula to solve equation (2):

$\sin(x) = \dfrac{-2 \pm \sqrt{12}}{4} = \dfrac{-1 \pm \sqrt{3}}{2}$.

Case 2a: $\sin(x) = \dfrac{-1+\sqrt{3}}{2}$.

It is noted that $0 < \dfrac{-1+\sqrt{3}}{2} < 1$ therefore x lies in either quadrant 1 or quadrant 2 of the unit circle.

Quadrant 1 solution:

$x = \sin^{-1}\left(\dfrac{\sqrt{3}-1}{2}\right) + 2n\pi,\ n \in Z$.

Quadrant 2 solution:

$x = \pi - \sin^{-1}\left(\dfrac{\sqrt{3}-1}{2}\right) + 2n\pi,\ n \in Z$

(by symmetery).

Case 2b: $\sin(x) = \dfrac{-1-\sqrt{3}}{2}$.

It is noted that $\dfrac{-1-\sqrt{3}}{2} < -1$ therefore $\sin(x) = \dfrac{-1-\sqrt{3}}{2}$ has no real solutions and is rejected.

Method 2:

Substitute

- $\cot(2x) = \dfrac{1}{\tan(2x)} = \dfrac{1-\tan^2(x)}{2\tan(x)}$
- $1 + \tan^2(x) = \sec^2(x) \Rightarrow \sec(x) = \pm\sqrt{1+\tan^2(x)}$

(from the VCAA formula sheet):

$\dfrac{1-\tan^2(x)}{2\tan(x)} = \pm\sqrt{1+\tan^2(x)}$

$\Rightarrow \left(\dfrac{1-\tan^2(x)}{2\tan(x)}\right)^2 = 1 + \tan^2(x)$

$\Rightarrow 1 - 2\tan^2(x) + \tan^4(x) = 4\tan^2(x)\big(1+\tan^2(x)\big)$

$\Rightarrow 3\tan^4(x) + 6\tan^2(x) - 1 = 0$

which is a quadratic equation in $\tan^2(x)$.

From the quadratic formula:

$\tan^2(x) = \dfrac{-6 \pm \sqrt{48}}{6} = \dfrac{-3 \pm 2\sqrt{3}}{3}$.

Case 1: $\tan^2(x) = \dfrac{-3-2\sqrt{3}}{3}$.

It is noted that $\dfrac{-3-2\sqrt{3}}{3} < 0$ therefore $\tan^2(x) = \dfrac{-3-2\sqrt{3}}{3}$ has no real solution and is rejected.

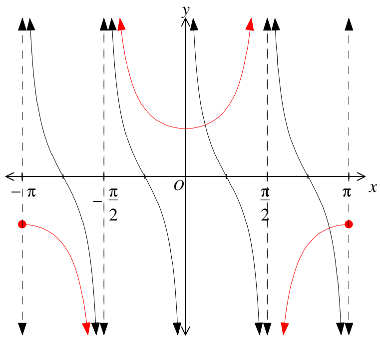
Case 2: $\tan^2(x) = \dfrac{-3+2\sqrt{3}}{3} \qquad \Rightarrow \tan(x) = \pm\sqrt{\dfrac{-3+2\sqrt{3}}{3}}$

$\Rightarrow x = \tan^{-1}\left(\pm\sqrt{\dfrac{2\sqrt{3}-3}{3}}\right) + n\pi,\ n \in Z$

$\Rightarrow x = \pm\tan^{-1}\left(\sqrt{\dfrac{2\sqrt{3}-3}{3}}\right) + n\pi$

(since $\tan^{-1}(\pm w) = \pm\tan^{-1}(w)$).

b. • Draw the graphs of $y = \cot(2x)$ and $y = \sec(x)$ on the domain $x \in [-\pi,\ \pi]$:



From **part a.** it is known that the graphs intersect at

$x = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$ and $x = \pi - \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$.

• By inspection of the graphs it is seen that $\cot(2x) < \sec(x)$ over the interval $x \in [-\pi,\ \pi]$ when

$$x \in \left(-\frac{\pi}{2},\ 0\right) \cup \left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right),\ \frac{\pi}{2}\right) \cup \left(\pi - \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right),\ \pi\right).$$