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CLASSROOM NOTE

When probability trees don't work

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Tree diagrams arise naturally in courses on probability at high school or university, even at an elementary level. Often they are used to depict outcomes and associated probabilities from a sequence of games. A subtle issue is whether or not the Markov condition holds in the sequence of games. We present two examples that illustrate the importance of this issue. Suggestions as to how these examples may be used in a classroom are offered.

Keywords: Markov condition; conditional probability; teaching probability; simulation

1. Introduction

Tom and Jerry play a sequence of games, and each game will be won by one player and lost by the other. If Jerry wins a game, then the probability that Tom wins the next game is p_1. If Tom wins a game, then the probability that Tom wins the next game is p_2. What is the probability, p_0, that Tom wins the third game given that he won the first game?

The answer depends on the game! We will substantiate this claim by presenting two games that lead to different answers.

2. Game 1

The first game is elementary and is based on moving around on a graph with three vertices. The graph is depicted in Figure 1. Let $V = \{0, T, J\}$ be the set of three vertices. At any stage, landing on the vertex T indicates that Tom wins, and landing on the vertex J indicates that Jerry wins; we use the vertex 0 only for starting the game.

Start at 0; toss a fair coin; if the coin shows Heads, move to T, and Tom wins the first game; otherwise move to J, and Jerry wins the first game.

In subsequent games we use only T and J, and moves are decided by tossing weighted coins. If we are on T, toss a weighted coin, then either move to J with probability α, or stay at T with probability $1 - \alpha$. If we are on J, toss a weighted coin, then either move to T with probability β, or stay at J with probability $1 - \beta$.

This game satisfies the requirements of the opening question with $p_1 = \beta$ and $p_2 = 1 - \alpha$.

The tree diagram in Figure 2 indicates the possible outcomes of the first three games where Tom won the first game. The probability that Tom wins the third game given that Tom won the first game is given by $(1 - \alpha)^2 + \alpha\beta$.

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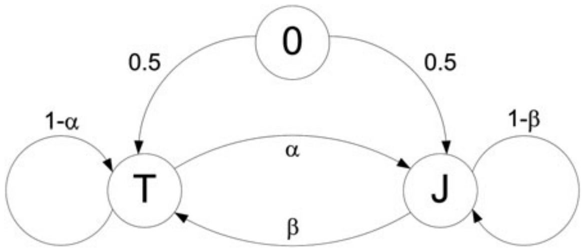

Figure 1. Graph for game 1.

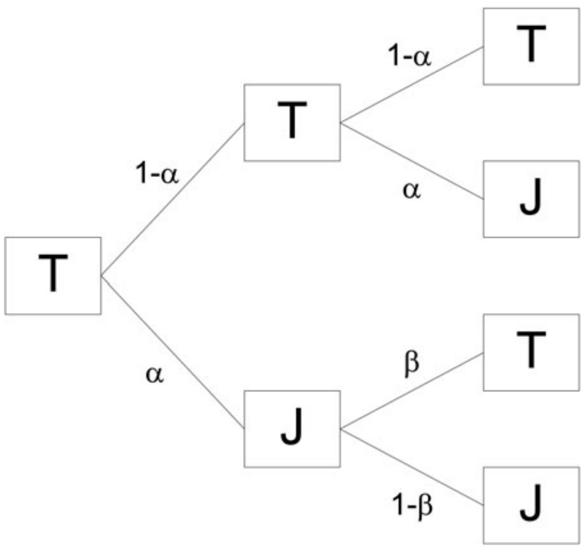

Figure 2. Tree diagram for game 1.

In the notation of the opening question, the probability that Tom wins the third game given that Tom won the first game is

$$p_0 = p_2^2 + p_1(1 - p_2).$$

3. Game 2

The second game is more complicated than the first. It stems from ideas in the paper [1] and references therein.

Suppose that we have a coin for which $P(\text{Head}) = p$ and $P(\text{Tail}) = q$ where $p + q = 1$ and $p \neq 0.5$. Toss the coin many times. Define the random variables Y_n ($n = 0, 1, \ldots$) by $Y_n = +1$ if the coin shows Heads on the n-th throw, and $Y_n = -1$ if the coin shows Tails on the n-th throw.

Define $X_n = Y_{n-1}Y_n$. Note that $X_n = -1$ or $+1$. We say that Jerry wins the n-th game if $X_n = -1$, otherwise Tom wins the n-th game.

We use the argument in [1] and obtain the following:

$$\begin{aligned}
p_1 &= P(\text{Tom wins the } n\text{-th game}|\text{Jerry won the } (n-1)\text{-th game})\\
&= P(X_n = +1|X_{n-1} = -1)\\
&= P(Y_{n-1}Y_n = +1|Y_{n-2}Y_{n-1} = -1)\\
&= P(Y_{n-1}Y_n = +1 \text{ and } Y_{n-2}Y_{n-1} = -1)/P(Y_{n-2}Y_{n-1} = -1)\\
&= (qp^2 + pq^2)/(2pq)\\
&= 0.5.
\end{aligned}$$

$$\begin{aligned}
p_2 &= P(\text{Tom wins the } n\text{-th game}|\text{Tom won the } (n-1)\text{-th game })\\
&= P(X_n = +1|X_{n-1} = +1)\\
&= P(Y_{n-1}Y_n = +1|Y_{n-2}Y_{n-1} = +1)\\
&= P(Y_{n-1}Y_n = +1 \text{ and } Y_{n-2}Y_{n-1} = +1)/P(Y_{n-2}Y_{n-1} = +1)\\
&= (p^3 + q^3)/(p^2 + q^2)\\
&= (p^2 - pq + q^2)/(p^2 + q^2).
\end{aligned}$$

Our opening question can be stated as follows: What is $p_0 = P(X_3 = +1|X_1 = +1)$? Bearing in mind that Y_0, Y_1, Y_2 and Y_3 are independent random variables, we have

$$\begin{aligned}
p_0 &= P(X_3 = 1|X_1 = 1)\\
&= P(Y_2Y_3 = 1|Y_0Y_1 = 1)\\
&= P(Y_2Y_3 = 1)\\
&= P(Y_2 = 1, Y_3 = 1) + P(Y_2 = -1, Y_3 = -1)\\
&= p^2 + q^2.
\end{aligned}$$

It is complicated to express p_0 in terms of p_1 and p_2 in this case. However, one can then check by calculation that, if $p \neq 0.5$ (say $p = 0.2$), then

$$p_0 \neq p_2^2 + p_1(1 - p_2).$$

Hence the answer to our opening question in this sequence of games is different from the answer in the first sequence of games.

4. Conclusion

We began this article with a question which one may encounter in a course on elementary probability. We have presented two different games that satisfied the requirements of the question–and they led to different answers. The question, as stated in the opening, is not well posed. Something is missing.

Normally, students would be expected to approach the problem according to the method in Game 1. However, this requires the assumption that the result of any game depends only on the result on the previous game; the Markov condition must hold, and this is what is missing in the statement of the problem. By the construction of Game 1, we guaranteed that

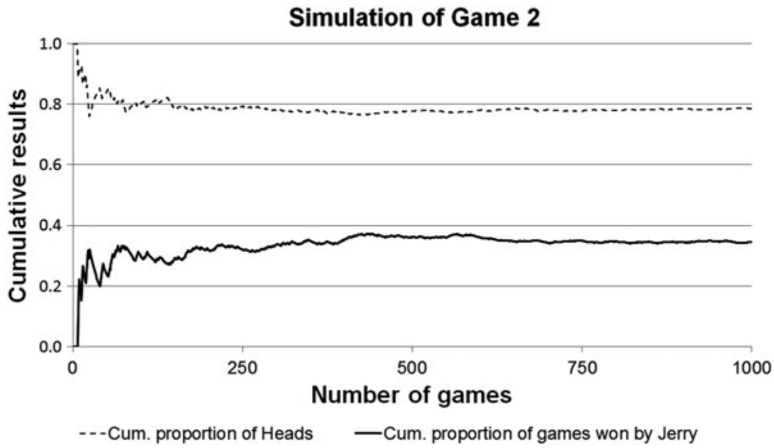

Figure 3. Results of simulation of game 2 with $p = P(Heads) = 0.8$.

the Markov condition holds, and the usual approach using a tree diagram is appropriate. Game 2 is non-Markovian. A tree diagram is not appropriate.

There are various ways in which the findings above can be used in the classroom.

In a class of elementary probability, such as in a high school, the teacher should be aware of this issue. An exercise or examination question along the lines of our opening question should be framed in such a way that the game is Markovian.

Sequences of many well known games are not likely to be Markovian. Discussing the following question in a class at university may be fruitful. Is it likely that a sequence of games of tennis between two players is Markovian or not? After considering this question for several different sports, students will appreciate the strength of the Markov condition. This will be useful for prospective school teachers who may have to teach probability during their career. Furthermore, it encourages students to make connections between what they are learning in class and the rest of the world.

A group project could be developed around reviewing text books that deal with probability trees. Do the sequences of games or other events described in the exercises satisfy the Markov condition, and hence justify the use of a probability tree? Teams of students could examine books and develop a group report on their findings. This exercise would also be useful for prospective teachers.

In a more advanced class, such as one for those majoring in mathematics or engineering, students might be asked to read this paper and present it to the class. Students might enjoy writing a program to simulate Game 2. We simulated sequences of 1000 trials of Game 2 for various values of $p = P(\text{Head})$ and recorded, for each $n = 1, 2, \ldots, 1000$, the cumulative proportion of Heads and the cumulative proportion of games won by Jerry. The results are illustrated in Figure 3. Such experiments will generate questions about, for example, the rate of convergence which go to the heart of limit theorems in probability.

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[1] Chan KC, Lenard CT, Mills TM. On Markov chains. Math Gaz. 2013;97(540):515–520.