ON SOLVING THE EQUATION $f(x) = f^{-1}(x)$

Ng Wee Leng and Ho Foo Him present a detailed set of arguments

Anecdotal evidence suggests that in solving the equation $f(x) = f^{-1}(x)$, where f is a given function whose inverse function f^{-1} exists, many students have been taught to solve, instead, the equation $f(x) = x$. No convincing explanation on why this method works, if at all, has been given by many of the teachers who use this approach. As a matter of fact, $f(x) = x$ is a sufficient, but not necessary, condition for $f(x) = f^{-1}(x)$.

In this article, we will discuss the concepts, and misconceptions, related to the above mathematical problem, and suggest how teachers might better explain, possibly with the aid of technology, various methods of solving the equation $f(x) = f^{-1}(x)$.

Relevant Results

We begin with four results related to solving the equation $f(x) = f^{-1}(x)$.

Result 1

If a function f is increasing in the domain D on which its inverse function f^{-1} exists, and a lies in D, then $f(a) = f^{-1}(a)$ if and only if $f(a) = a$.

Result 2

If a function f is decreasing in the domain D, then the equation $f(x) = x$ has at most one solution.

Result 3

If a function f is defined in the domain D on which its inverse function f^{-1} and the composite function ff exist, then $f(x) = f^{-1}(x)$ if and only if $ff(x) = x$.

Result 4

If a function f is defined in the domain D on which its inverse function f^{-1} exists, and a lies in D, then $f(a) = f^{-1}(a)$ if and only if there exists b which lies in D such that $f(a) = b$ and $f(b) = a$.

It is easy to understand why Result 1 holds true; if $f(a) = f^{-1}(a)$ and say $f(a) > a$, then since f is increasing on D, we have $f(a) > a = f(f(a)) > f(a) = f^{-1}(a)$, a contradiction.

Result 2 is true for otherwise, say $f(a) = a, f(b) = b$ and $a < b$, then as f is decreasing, $f(a) > f(b)$ which implies that $a > b$, a contradiction.

On the other hand, Result 3 follows from the properties of an inverse function.

Lastly, Result 4 is true because if $f(a) = f^{-1}(a)$ where a lies in the domain D on which both the functions f and f^{-1} are defined, then we let $b = f^{-1}(a)$, which lies in D, and $f(b) = a$, as required.

Results 1 and 2 show that while the method of solving the equation $f(x) = x$ for all the solutions of the equation $f(x) = f^{-1}(x)$ will work if f is increasing, in cases where the given function is decreasing in the entire domain, or in an interval which is a subset of the domain, solving $f(x) = x$ may not yield all the solutions of $f(x) = f^{-1}(x)$. As expressing the inverse function of a bijective function f in closed form might not always be possible, defining the composite function ff explicitly is, however, straightforward. Result 3 thus provides us with a method of solving the equation $f(x) = f^{-1}(x)$ completely. Note that only solutions which lie in the domain on which ff is defined are admissible. If the inverse function of a given function can be expressed explicitly, we can of course solve the equation $f(x) = f^{-1}(x)$, bearing in mind that only solutions lie in the domain on which both f and f^{-1} are defined are admissible.

Result 4 provides a geometrical interpretation of the solutions of the equation $f(x) = f^{-1}(x)$ as follows. Each solution is either the abscissa of a point of intersection of the graph of f with the line $y = x$, which is the line of symmetry of the graphs of f and f^{-1}, or the abscissa of a point on the graph of f of which another point on the graph of f is the 'mirror image' about the line $y = x$.

Result 4 also suggests that solving the equations $f(a) = b$, and $f(b) = a$ simultaneously for a and b will yield all the solutions of the equation $f(x) = f^{-1}(x)$.

Teachers could explain these concepts by using suitable examples to show the various situations with the aid of graphing tools. What follows is an illustration of this approach. The graphing tool we used is *GeoGebra* which is free software.

Some Examples to Illustrate the Concepts and Misconceptions

Given a function g, we shall denote its domain as D_g and its range as R_g.

Example 1

The function f is given by $f(x) = \sqrt{7-3x}, x \leq \frac{7}{3}$.

Find the solution(s) of the equation $f(x) = f^{-1}(x)$. First note that f is decreasing on its domain $D_f = (-\infty, \frac{7}{3}]$ and $R_f = [0, \infty)$. Thus both f and f^{-1} are defined on $[0, \frac{7}{3}]$. Let $y = \sqrt{7-3x}$. Then $y^2 = 7 - 3x$ and so $x = \frac{7-y^2}{3}$.

It follows that $f^{-1}(x) = \frac{7-x^2}{3}, x \geq 0$.

Let's first solve the equation $f(x) = x$ which is equivalent to $x^2 + 3x - 7 = 0$ and whose solutions are $x = \frac{-3 \pm \sqrt{37}}{2}$. Since f and f^{-1} are defined on $[0, \frac{7}{3}]$, we accept only $x = \frac{-3+\sqrt{37}}{2} \approx 1.54$ as a solution of the equation $f(x) = f^{-1}(x)$.

Have we found all the solutions of $f(x) = f^{-1}(x)$? Let's graph the functions f and f^{-1} with the aid of a graphing tool. From the graph above - see Figure 1, we can see that there are 3 points of intersections, namely $A(1.54, 1.54)$, $B(2, 1)$ and $C(1, 2)$. That is, the solutions to the equation $f(x) = f^{-1}(x)$ are $x = 1$, $x = 1.54$, and $x = 2$. In other words, solving $f(x) = x$ does not yield all the solutions of $f(x) = f^{-1}(x)$. Note that point A lies on the line $y = x$ while points B and C are reflections of each other about the line $y = x$, as described in Result 4.

To find all the solutions of the equation $f(x) = f^{-1}(x)$ we could solve the equation directly in this case since f^{-1} can be explicitly defined. That is, we solve the equation

$\sqrt{7-3x} = \frac{7-x^2}{3}$ which leads to the quartic equation $x^4 - 14x^2 + 27x - 14 = 0$. Factorising the quartic expression we obtain

$(x - 1)(x - 2)(x^2 + 3x - 7) = 0$.

Admitting only values in the interval $[0, \frac{7}{3}]$, the required solutions are $x = 1, 2,$ and $\frac{-3+\sqrt{37}}{2}$

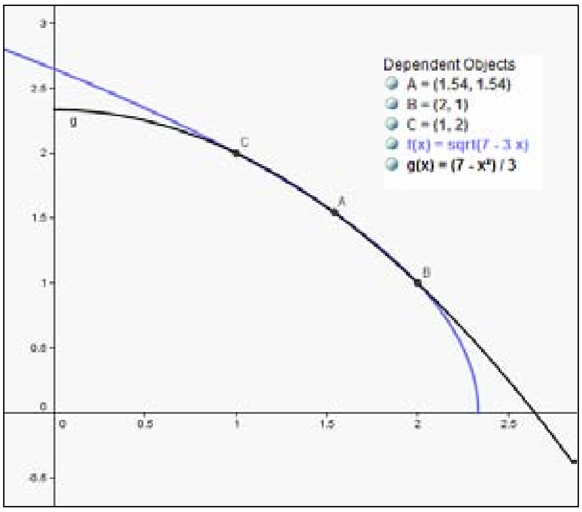

Figure 1: Graphs of $f(x) = \sqrt{7-3x}$ and its inverse function intersect at 3 points, *A*, *B* and *C*.

Alternatively, we solve the equation $ff(x) = x$. It is easy to see that $D_{ff} = [\frac{14}{27}, \frac{7}{3}]$, and so we solve $\sqrt{7 - 3\sqrt{7-3x}} = x$,

for x in the interval $[\frac{14}{27}, \frac{7}{3}]$, which, as before, leads to the equation $\sqrt{7-3x} = \frac{7-x^2}{3}$ and the solutions as obtained above.

We could also solve the equations $\sqrt{7-3a} = b$ and $\sqrt{7-3b} = a$ simultaneously for a and b.

Note that $b^2 = 7 - 3a$, and $a^2 = 7 - 3b$, implying $b^2 - a^2 = 3(b - a)$, or equivalently,

$(b - a)(b + a - 3) = 0$. If $a = b$, then $a^2 = 7 - 3a$ whose solution is $\frac{-3+\sqrt{37}}{2}$, since $a > 0$.

If $a + b = 3$, then $3 - a = \sqrt{7-3a}$, and thus $9 - 6a + a^2 = 7 - 3a$. Hence, $a^2 - 3a + 2 = 0$, that is $(a - 1)(a - 2) = 0$, and therefore the roots are $a = 1$, or $a = 2$.

Example 2

Let $f(x) = \frac{x^3-1}{5x^3-1}$ for $x \neq \sqrt[3]{\frac{1}{5}}$. Find the solution(s) of the equation $f(x) = f^{-1}(x)$.

With the aid of a graphing tool, we graph the function f as shown below - see figure 2, and observe that the graph of f does not intersect the line $y = x$.

Does that mean that the equation $f(x) = f^{-1}(x)$ has no solution? Certainly not! As shown in the graph, points A and A', and respectively B and B', are reflections of each other about the line $y = x$.

Thus there are four solutions to the equation $f(x) = f^{-1}(x)$.

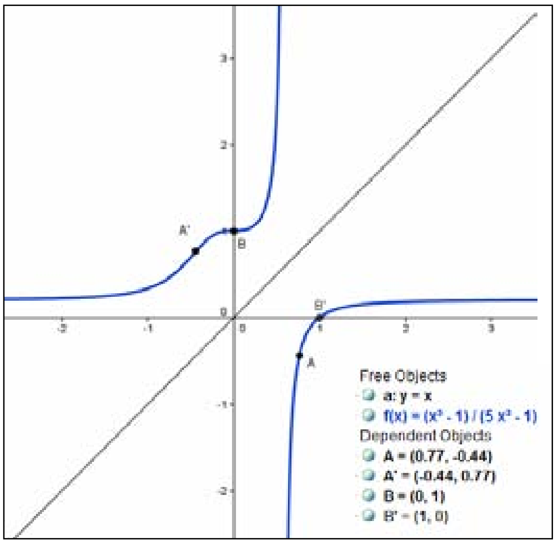

Figure 2 : The graph of $f(x) = \dfrac{x^3 - 1}{5x^3 - 1}$ does not intersect the line $y = x$

To further verify that $f(x) = f^{-1}(x)$ has four solutions, we also graph the function f^{-1} which is given by $f^{-1}(x) = \sqrt[3]{\dfrac{x-1}{5x-1}}$. Indeed, the graphs of f and f^{-1}, as shown in figure 3, intersect at four points. To find the exact solutions of $f(x) = f^{-1}(x)$, we can solve the equation directly, which entails solving the equation

$(x^3 - 1)^3 (5x - 1) - (5x^3 - 1)^3 (x - 1) = 0$

or, the equivalent equation

$120x^{10} - 124x^9 - 60x^7 - 72x^6 - 12x^3 + 4x = 0$.

Factorising the polynomial in x yields the equation

$4x(x - 1)(3x^2 - x - 1)(10x^6 + 3x^5 + 4x^4 - 3x^3 + x^2 + 1) = 0$

The roots of $3x^2 - x - 1 = 0$ are $\dfrac{1 \pm \sqrt{13}}{6}$ whereas $10x^6 + 3x^5 + 4x^4 - 3x^3 + x^2 + 1 = 1$ has no real roots. Hence the four solutions are $x = 0$, $x = 1$, and

$x = \dfrac{1 \pm \sqrt{13}}{6}$.

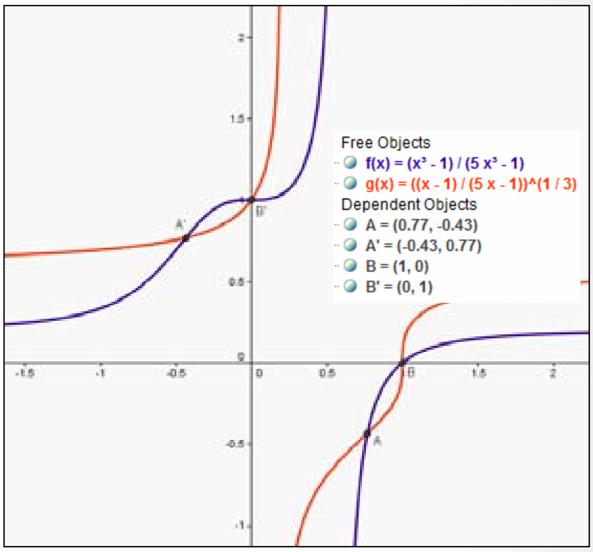

Figure 3: The graphs of $f(x) = \dfrac{x^3 - 1}{5x^3 - 1}$ and its inverse function intersect at four points.

We can also solve the equation $ff(x) = x$ where

$ff(x) = \dfrac{x^3(31x^6 - 18x^3 + 3)}{30x^9 - 15x^6 + 1}$.

To this end, we solve the equation

$x(30x^9 - 31x^8 - 15x^6 + 18x^5 - 3x^2 + 1) = 0$ which is equivalent to

$x(x - 1)(3x^2 - x - 1)(10x^6 + 3x^5 + 4x^4 - 3x^3 + x^2 + 1) = 0$

On the other hand, the equation $f(x) = x$ which leads to the equation $5x^4 - x^3 - x + 1 = 0$ has no real roots.

As in Example 1, we could also solve the equation $f(a) = b$ and $f(b) = a$ simultaneously for a and b. To this end, we first let

$b = \dfrac{a^3 - 1}{5a^3 - 1}$ and $a = \dfrac{b^3 - 1}{5b^3 - 1}$.

Then $a^3 - 1 = b(5a^3 - 1)$ and $b^3 - 1 = a(5b^3 - 1)$.

Thus $a^3 - b^3 = 5ba^3 - b - 5ab^3 + a$

$= 5ab(a^2 - b^2) + a - b$ or, equivalently,

$(a - b)(a^2 + ab + b^2 - 5ab(a + b) - 1) = 0$,

implying

$a = b$ or $a^2 + ab + b^2 - 5ab(a + b) - 1 = 0$.

If $a = b$ then $a^3 - 1 = a(5a^3 - 1)$ and thus

$5a^4 - a^3 - a + 1 = 0$ which has no real roots.

If $a^2 + ab + b^2 - 5ab(a + b) - 1 = 0$, we substitute

$b = \dfrac{a^3 - 1}{5a^3 - 1}$ into $a^2 + ab + b^2 - 5ab(a + b) - 1 = 0$

and obtain the equation

$a(6a^5 - 5a^4 - a^3 - 2a^2 + a + 1) = 0.$

Factorising the polynomial in a, we obtain the equation $a(a - 1)(2a^2 + a + 1)(3a^2 - a - 1) = 0.$

The roots of the equation $3a^2 - a - 1$ are $\frac{1 \pm \sqrt{13}}{6}$, whereas $2a^2 + a + 1 = 0$ has no real roots. Hence the four solutions are $a = 0$, $a = 1$, and $a = \frac{1 \pm \sqrt{13}}{6}$.

Example 3

Let $f(x) = \frac{x-2}{x-1}, x \neq 1.$ Find the solution(s) of the equation $f(x) = f^{-1}(x)$.

From the graph below - see Figure 4, we observe that the function f is self-inverse. Indeed, it is easy to see that

$$y = \frac{x-2}{x-1} \Rightarrow yx - y = x - 2, \text{ giving}$$

$$x = \frac{y-2}{y-1}. \text{ Thus } f^{-1}(x) = \frac{x-2}{x-1}, x \neq 1.$$

Therefore the solution set of the equation $f(x) = f^{-1}(x)$ is the set of all real numbers excluding 1. The equation $ff(x) = x$ will also yield the same solution set since for every real number x except 1, $ff(x) = x$. However, the equation $f(x) = x$ will yield no real roots as the graph of f does not intersect the line $y = x$

The graph of $f(x) = \frac{x-2}{x-1}, x \neq 1.$ is the same as that of its inverse function.

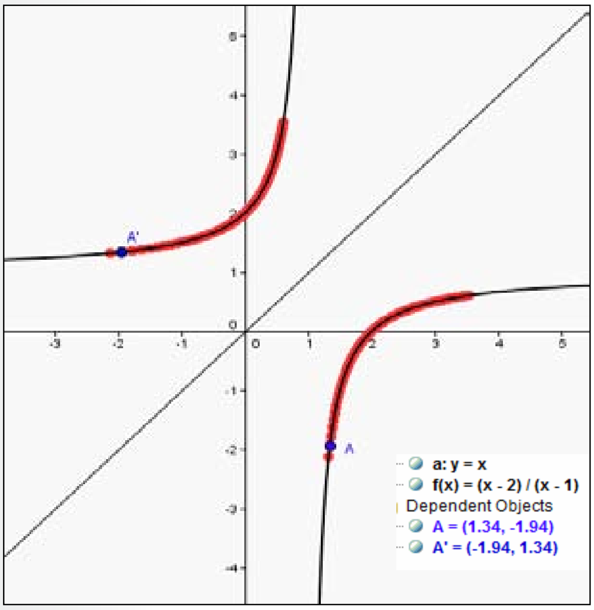

Figure 4: Point A traces out point A'

With the help of a graphing tool, we could trace the graph of f to verify that for every point (a, b) on the graph of f such that $a \neq 1$ and $b \neq 1$, its reflection (b, a) about the line $y = x$ also lies on the graph of f. It then follows from Result 4 that the solution set of the equation $f(x) = f^{-1}(x)$ is the real line excluding 1.

Conclusion

The roots of the equation $f(x) = f^{-1}(x)$, if they exist, can be obtained by solving $f(x) = x$ if f is increasing in a domain on which its inverse function exists. A solution may not exist for example, when $f(x) = x + e^x$ which is increasing for all real x, but whose graph lies entirely above the line $y = x$, or there could be one or more solutions for example, the graph of f given by $f(x) = 2x + e^x$ cuts the line $y = x$ exactly once, whereas that of g given by $g(x) = 3 - x + e^{x-2}$ intersects the line $y = x$ exactly twice.

From the three examples given in the previous section, we know that the method of solving $f(x) = x$ fails to obtain all the solutions of the equation $f(x) = f^{-1}(x)$ if the function f is a self-inverse function, or if there exist real numbers a and b, $a \neq b$, such that the points (a, b), and (b, a) both lie on the graph of f.

On the other hand, solving the equation $ff(x) = x$ will always yield all the solutions of the equation $f(x) = f^{-1}(x)$, and that given the expression for $f(x)$, $ff(x)$ can always be expressed explicitly. Exact solution(s) can be found if the former equation leads to a polynomial equation with radical roots. Otherwise, with the aid of a graphing tool, numerical solutions of the equation can always be found.

Finally, to solve the equation $f(x) = f^{-1}(x)$, we first determine whether the function f is increasing. If it is, we solve the equation $f(x) = x$ for x in the domain on which both f and f^{-1} are defined. If it is not, we solve the equation $ff(x) = x$ for x in the domain on which the composite function ff is defined. We can also solve the equations $f(a) = b$ and $f(b) = a$ for a and b simultaneously as the solution set of $f(x) = f^{-1}(x)$ is the set of all such a, b. However, in most cases finding all such a and b analytically may not be easy, unless a computer algebra system is available.

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