VCAA 2020 SPECIALIST MATHEMATICS EXAMINATION 2

Question 3

Let $g(x) = x^n e^{-x}$, where $n \in Z$.

e. **ii.** Complete the following table by stating the value(s) of n for which the graph of $g(x)$ has the given number of points of inflection. 2 marks

Number of points of inflection	Value(s) of n (where $n \in Z$)
0	
1	
2	
3	

Solution

• *Candidates* for points of inflection are solutions to $g''(x) = 0$:

```
In[267]:= g[x_, n_] := x^n * Exp[-x]

In[278]:= Simplify[D[g[x, n], {x, 2}]]
Out[278]= e^-x x^(-2+n) (n^2 + x^2 - n (1 + 2 x))

In[279]:= Reduce[e^-x x^(-2+n) (n^2 + x^2 - n (1 + 2 x)) == 0, x]
Out[279]= (Re[n] > 2 && x == 0) || (x ≠ 0 && (x == -√n + n || x == √n + n))
```

It is noted that these solutions only exist for $n > 0$ so only $n > 0$ is investigated.

A solution to $g''(x) = 0$ corresponds to a point of inflection of $g(x)$ only if it corresponds to a **turning point** of $g'(x)$. The 'double derivative' test will therefore be applied to $g'(x)$.

Case 1: $x = n + \sqrt{n}$.

Consider the value of $g'''\left(n + \sqrt{n}\right)$:

```
In[281]:= Simplify[D[g[x, n], {x, 3}]]
Out[281]= e^-x x^(-3+n) (n^3 - x^3 - 3 n^2 (1 + x) + n (2 + 3 x + 3 x^2))

In[285]:= h[x_, n_] := n^3 - x^3 - 3 n^2 (1 + x) + n (2 + 3 x + 3 x^2)

In[286]:= Simplify[h[√n + n, n]]
Out[286]= 2 (n + n^(3/2))
```

$g'''\left(n + \sqrt{n}\right) \neq 0$ therefore $g'(x)$ has a turning point at $x = n + \sqrt{n}$ therefore $g(x)$ has a point of inflection at $x = n + \sqrt{n}$.

Case 2: $x = n - \sqrt{n}$.

Consider the value of $g'''\left(n - \sqrt{n}\right)$:

```
In[287]:= Simplify[h[-√n + n, n]]
Out[287]= -2 (-1 + √n) n
```

Case 2a: $n = 1 \quad \Rightarrow x = 0$.

```
In[362]:= Simplify[D[g[x, 1], {x, 2}]] /. x → 0
Out[362]= -2
```

$g''(0) \neq 0$ therefore $g(x)$ does not have a point of inflection when $n = 1$.

Case 2b: $n > 1$.

$g'''\left(n - \sqrt{n}\right) \neq 0$ therefore $g'(x)$ has a turning point at $x = n - \sqrt{n}$ therefore $g(x)$ has a point of inflection at $x = n - \sqrt{n}$.

Case 3: $x = 0$ and $n > 2$.

Consider the value of $g'''(0)$:

```
In[295]:= Simplify[D[g[x, n], {x, 3}]] /. x → 0
```

```
Out[295]= 0^(-3+n) (2 n - 3 n^2 + n^3)
```

Case 3a: $n = 3$.

```
In[296]:= Simplify[D[g[x, 3], {x, 3}]] /. x → 0
```

```
Out[296]= 6
```

$g'''(0) \neq 0$ therefore $g'(x)$ has a turning point at $x = 0$ therefore $g(x)$ has a point of inflection at $x = 0$.

Case 3b: $n > 3$.

$g'''(0) = 0$ therefore the 'double derivative' test is inconclusive and so further investigation is required.

```
In[307]:= Refine[Simplify[D[g[x, n], {x, 2}]] /. x → 0, {n ∈ Integers, n > 3}]
```

```
Out[307]= 0
```

$g''(0) = 0$ for $n > 3$ therefore there is a potential point of inflection at $x = 0$. The sign of $g''(x)$ on either side of $x = 0$ must be investigated. If there is a change in sign then there is a change in concavity and therefore a point of inflection at $x = 0$.

```
In[347]:= Simplify[D[g[x, n], {x, 2}]] /. x → 1
```

$$\text{Out[347]=}\ \frac{1 - 3n + n^2}{e}$$

```
In[353]:= Reduce[{n^2 - 3*n + 1 > 0, n > 2}, {n ∈ Integers}] // N
```

```
Out[353]= n ∈ ℤ && n ≥ 3.
```

```
In[348]:= Simplify[D[g[x, n], {x, 2}]] /. x → -1
```

$$\text{Out[348]=}\ (-1)^{-2+n}\, e\, (1 + n + n^2)$$

x	-1	0	1
$g''(x)$	$(-1)^{n-2}(n^2+n+1)$ $\begin{cases} >0 & n \text{ even} \\ <0 & n \text{ odd} \end{cases}$	0	$\frac{n^2-3n+1}{e} > 0$

Therefore the sign of $g''(x)$ only changes on either side of $x = 0$ when $n > 3$ is odd.

Therefore $g(x)$ has a point of inflection at $x = 0$ only when $n > 2$ is odd.

$g(x)$ does **not** have a point of inflection at $x = 0$ when $n > 2$ is even.

Summary:

Case 1: Point of inflection at $x = n + \sqrt{n}$ if $n \geq 1$.

Case 2: Point of inflection at $x = n - \sqrt{n}$ if $n \geq 2$.

Case 3: Point of inflection at $x = 0$ if $n \geq 3$ and n odd.

Number of points of inflection	**Value(s) of n where $n \in Z$**
0	$n \leq 0$
1	$n = 1$
2	$n \geq 2$ and n is **even**
3	$n \geq 3$ and n is **odd**

```
In[383]:= Plot[{g[x, 1], g[x, 2], g[x, 3]}, {x, -0.5, 5},
  PlotStyle → {Red, Blue, Green}]
```

Out[383]=