MATHEMATICAL METHODS EXAM 2

Section B - Question 20

SOLUTION AND DISCUSSION

Solution:

- $f(x)$ is periodic. An obvious period satisfying $f(x) = f(x+h)$ for all $h \in Z$ is 1.

  For example, if $f(x) = f(x+3)$ and the period is 1 then:

  $f(x) = f(x+1)$. $f(x+1) = f([x+1]+1) = f(x+2)$. $f(x+2) = f([x+2]+1) = f(x+3)$.

  It therefore follows that $f(x) = f(x+3)$ ✓

Note: Other periods are possible and are of the form $\frac{1}{n}$ where $n \in Z^+$. The period of 1 is the largest and most obvious of these and corresponds to $n = 1$.

- Check if there is an option consistent with $f(x)$ having a period of 1:

$f(x) = \cos(ax)$ therefore $\frac{2\pi}{a} = 1 \Rightarrow a = 2\pi$.

Therefore $f(x) = \cos(2\pi x)$.

- It is required that $-1 \le \log_2\left(f(x)\right) \le 0$:

$-1 \le \log_2\left(\cos(2\pi x)\right) \le 0 \qquad \Rightarrow \frac{1}{2} \le \cos(2\pi x) \le 1$

and only option B satisfies this inequality.

Answer: B.

Discussion: There are other possible intervals for D when the period is 1.

$\cos(2\pi x) = 1$

$\Rightarrow 2\pi x = 2m\pi$ where $m \in Z$

$\Rightarrow x = m$.

$\cos(2\pi x) = \frac{1}{2}$

Case 1: $2\pi x = \frac{\pi}{3} + 2m\pi \qquad \Rightarrow x = \frac{1}{6} + m = \frac{6m+1}{6}$.

Case 2: $2\pi x = -\frac{\pi}{3} + 2m\pi \qquad \Rightarrow x = -\frac{1}{6} + m = \frac{6m-1}{6}$.

Therefore the maximal domain is the union of all intervals of the form $\left[\frac{6m-1}{6},\ m\right] \cup \left[m,\ \frac{6m+1}{6}\right]$, where $m \in Z$, and any interval having the form $\left[\frac{6m-1}{6},\ m\right]$ or $\left[m,\ \frac{6m+1}{6}\right]$, where $m \in Z$, is a possible interval for D.

So there are an infinite number of possible intervals and **option B** corresponds to the particular interval given by $\left[m,\ \frac{6m+1}{6}\right]$ and $m = 1$. No other option corresponds to an interval having either of the above forms.

General maximal domain:

- In general the possible periods of $f(x)$ are of the form $\frac{1}{n}$ where $n \in Z^+$.

For example, if $f(x) = f(x+3)$ and the period is $\frac{1}{2}$ then:

$$f(x) = f\left(x + \frac{1}{2}\right). \qquad f\left(x + \frac{1}{2}\right) = f\left(\left[x + \frac{1}{2}\right] + \frac{1}{2}\right) = f(x+1) \quad etc.$$

It therefore follows that $f(x) = f(x+3)$ ✓

- Let the period of $f(x)$ be $\frac{1}{n}$:

$f(x) = \cos(ax)$ therefore $\frac{2\pi}{a} = \frac{1}{n} \Rightarrow a = 2n\pi$.

Therefore $f(x) = \cos(2n\pi x)$.

- It is required that $-1 \le \log_2\left(f(x)\right) \le 0$:

$-1 \le \log_2\left(\cos(2n\pi x)\right) \le 0 \qquad \Rightarrow \frac{1}{2} \le \cos(2n\pi x) \le 1$.

$\cos(2n\pi x) = 1$

$\Rightarrow 2n\pi x = 2m\pi$, $m \in Z$

$\Rightarrow x = \frac{m}{n}$.

$\cos(2n\pi x) = \frac{1}{2}$

Case 1: $2n\pi x = \frac{\pi}{3} + 2m\pi \qquad \Rightarrow x = \frac{6m+1}{6n}$.

Case 1: $2n\pi x = -\frac{\pi}{3} + 2m\pi \qquad \Rightarrow x = \frac{6m-1}{6n}$.

Therefore the maximal domain is the union of all intervals of the form $\left[\frac{6m-1}{6n}, \frac{m}{n}\right] \cup \left[\frac{m}{n}, \frac{6m+1}{6n}\right]$ where $n \in Z^+$ and $m \in Z$.

There are an infinite number of possible intervals for D and these intervals depend on the period of $f(x)$.

Only **option B** offers an interval having one of these forms ($n = m = 1$). So there is only one correct option and that option is only correct when the period of $f(x)$ is 1. It could be argued that this question is defective inasmuch as there is only a correct option for a **specific period**.