MATHEMATICAL METHODS EXAM 2

Section B - Question 20

SOLUTION AND DISCUSSION

Solution:

- If $f(x) = f(x+h)$ for all $h \in Z$ then $f(x)$ is periodic and an obvious (and in fact largest) possible period is 1.

  For example, if $f(x) = f(x+3)$ and the period is 1 then:

  $f(x) = f(x+1)$. $\quad$ $f(x+1) = f([x+1]+1) = f(x+2)$. $\quad$ $f(x+2) = f([x+2]+1) = f(x+3)$.

  It therefore follows that $f(x) = f(x+3)$ ✓

- $f(x) = \cos(ax)$ and the period is 1 therefore $\dfrac{2\pi}{a} = 1 \Rightarrow a = 2\pi$.

Therefore $f(x) = \cos(2\pi x)$.

- It is required that $-1 \le \log_2\left(f(x)\right) \le 0$:

$-1 \le \log_2\left(\cos(2\pi x)\right) \le 0 \qquad \Rightarrow \dfrac{1}{2} \le \cos(2\pi x) \le 1$ and only Option B works.

Answer: B.

Discussion: Other possible intervals for *D* are possible:

$\cos(2\pi x) = 1$

$\Rightarrow 2\pi x = 2m\pi$ where $m \in Z$

$\Rightarrow x = m$.

$\cos(2\pi x) = \dfrac{1}{2}$

Case 1: $2\pi x = \dfrac{\pi}{3} + 2m\pi \qquad \Rightarrow x = \dfrac{1}{6} + m = \dfrac{6m+1}{6}$.

Case 1: $2\pi x = -\dfrac{\pi}{3} + 2m\pi \qquad \Rightarrow x = -\dfrac{1}{6} + m = \dfrac{6m-1}{6}$.

Therefore the maximal domain is the union of all intervals of the form $\left[\dfrac{6m-1}{6},\ m\right] \cup \left[m,\ \dfrac{6m+1}{6}\right]$ where $m \in Z$.

Any interval of the form either $\left[\dfrac{6m-1}{6},\ m\right]$ or $\left[m,\ \dfrac{6m+1}{6}\right]$ where $m \in Z$ is a possible interval for *D*.

So there are an infinite number of possible intervals and **option B** corresponds to the particular interval given by $\left[m,\ \dfrac{6m+1}{6}\right]$ and $m = 1$. None of the other intervals in the remaining options correspond to intervals of these forms.

General maximal domain:

- In general the possible periods of $f(x)$ are of the form $\dfrac{1}{n}$ where $n \in Z^+$.

(The largest period is 1 and corresponds to $n = 1$).

For example, if $f(x) = f(x+3)$ and the period is $\dfrac{1}{2}$ then:

$f(x) = f\left(x + \dfrac{1}{2}\right)$. $\qquad f\left(x + \dfrac{1}{2}\right) = f\left(\left[x + \dfrac{1}{2}\right] + \dfrac{1}{2}\right) = f(x+1)$ *etc.*

It therefore follows that $f(x) = f(x+3)$ ✓

- $f(x)=\cos(ax)$ and the period is $\frac{1}{n}$ therefore $\frac{2\pi}{a}=\frac{1}{n}\Rightarrow a=2n\pi$.

Therefore $f(x)=\cos(2n\pi x)$.

- It is required that $-1\le\log_2\left(f(x)\right)\le 0$:

$-1\le\log_2\left(\cos(2n\pi x)\right)\le 0 \qquad \Rightarrow \frac{1}{2}\le\cos(2n\pi x)\le 1$.

$\cos(2n\pi x)=1$

$\Rightarrow 2n\pi x=2m\pi$, $m\in Z$

$\Rightarrow x=\frac{m}{n}$.

$\cos(2n\pi x)=\frac{1}{2}$

Case 1: $2n\pi x=\frac{\pi}{3}+2m\pi \qquad \Rightarrow x=\frac{6m+1}{6n}$.

Case 1: $2n\pi x=-\frac{\pi}{3}+2m\pi \qquad \Rightarrow x=\frac{6m-1}{6n}$.

Therefore the maximal domain is the union of all intervals of the form $\left[\frac{6m-1}{6n},\ \frac{m}{n}\right]\cup\left[\frac{m}{n},\ \frac{6m+1}{6n}\right]$ where $n\in Z^+$ and $m\in Z$.

There are an infinite number of possible intervals for *D* and these intervals depend on the period of $f(x)$.