Proof of linear independence theorem:

Let the vectors $\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$, $\underset{\sim}{b} = b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k}$, $\underset{\sim}{c} = c_1 \underset{\sim}{i} + c_2 \underset{\sim}{j} + c_3 \underset{\sim}{k}$.

Let $\gamma \underset{\sim}{a} + \alpha \underset{\sim}{b} + \beta \underset{\sim}{c} = \underset{\sim}{0}$

$$\Rightarrow \gamma\left(a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}\right) + \alpha\left(b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k}\right) + \beta\left(c_1 \underset{\sim}{i} + c_2 \underset{\sim}{j} + c_3 \underset{\sim}{k}\right) = \underset{\sim}{0}.$$

Equate components:

$\underset{\sim}{i}$ -components: $\gamma a_1 + \alpha b_1 + \beta c_1 = 0$. …. (1)

$\underset{\sim}{j}$ -components: $\gamma a_2 + \alpha b_2 + \beta c_2 = 0$. …. (2)

$\underset{\sim}{k}$ -components: $\gamma a_3 + \alpha b_3 + \beta c_3 = 0$. …. (3)

Write equations (1), (2) and (3) as a matrix equation: $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly independent then the only solution to this system of equations is the trivial solution $\gamma = \alpha = \beta = 0$.

This only occurs if $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0$.

(The proof of this statement requires a significant amount of matrix theory that is beyond the scope of what can be taught at the Specialist Mathematics level).

Therefore $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly independent if $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0$.

Note: $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0$ corresponds to either an infinite number of solutions (which include the trivial solution $\gamma = \alpha = \beta = 0$) or no solution. The possibility of no solution is rejected because the trivial solution $\gamma = \alpha = \beta = 0$ always exists.