Note on the *Mathematica* answer:

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In[770]:= ToRadicals[Solve[{Cot[2 x] == Sec[x], 0 < x < Pi}, x]]
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$$\text{Out[770]=}\ \left\{\left\{x\to 2\,\text{ArcTan}\left[1+\sqrt{3}-\sqrt{3+2\sqrt{3}}\right]\right\},\left\{x\to 2\,\text{ArcTan}\left[1+\sqrt{3}+\sqrt{3+2\sqrt{3}}\right]\right\}\right\}$$

This answer is equivalent but different to the answers found using **Method 1** and **Method 2**. It appears to be a very peculiar answer but can be found in the following way (which a human would never do):

- Substitute $t=\tan\left(\frac{x}{2}\right)$ (the *Weierstrass substitution*)

into $\cot(2x)=\sec(x)$:

$$t=\tan\left(\frac{x}{2}\right)\quad\Rightarrow \sin(x)=\frac{2t}{t^2+1}\quad\text{and}\quad\cos(x)=\frac{1-t^2}{t^2+1}.$$

$$\tan\left(\frac{x}{2}\right)=t$$

$$\Rightarrow \sin\left(\frac{x}{2}\right)=\frac{t}{\sqrt{1+t^2}}\quad\text{and}\quad\cos\left(\frac{x}{2}\right)=\frac{1}{\sqrt{1+t^2}}.$$

Therefore:

$$\sin(x)=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)=\frac{2t}{t^2+1}.$$

$$\cos(x)=\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)=\frac{1}{t^2+1}-\frac{t^2}{t^2+1}=\frac{1-t^2}{t^2+1}.$$

- Therefore:

$$\tan(x)=\frac{2t}{1-t^2}$$

$$\Rightarrow \tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$$

$$=\frac{\frac{4t}{1-t^2}}{1-\left(\frac{4t}{1-t^2}\right)^2}\qquad=\frac{4t\left(1-t^2\right)}{\left(1-t^2\right)^2-4t^2}\qquad=\frac{4t\left(1-t^2\right)}{t^4-6t^2+1}.$$

- Therefore:

$$\cot(2x)=\sec(x)$$

$$\Rightarrow \frac{t^4-6t^2+1}{4t\left(1-t^2\right)}=\frac{1+t^2}{1-t^2}\qquad\Rightarrow t^4-6t^2+1=4t\left(1+t^2\right)$$

$$\Rightarrow t^4-4t^3-6t^2-4t+1=0.\qquad\text{.... (1)}$$

- Equation (1) is a *symmetric quartic equation*

  (a *symmetric quartic equation* has the form

  $$at^4+bt^3+ct^2+bt+a=0$$

  and is a solvable 'by hand' special case of a quartic equation).

The following standard approach is used to convert equation (1) into a quadratic equation:

$$t^4-4t^3-6t^2-4t+1=0$$

$$\Rightarrow t^2-4t-6-\frac{4}{t}+\frac{1}{t^2}=0,\quad t\neq 0$$

$$\Rightarrow \left(t^2+\frac{1}{t^2}\right)-4\left(t+\frac{1}{t}\right)-6=0.\qquad\text{.... }(1')$$

Substitute

$$w=t+\frac{1}{t}$$

$$\Rightarrow w^2=t^2+2+\frac{1}{t^2}\qquad\Rightarrow w^2-2=t^2+\frac{1}{t^2}$$

into equation $(1')$:

$$\left(w^2-2\right)-4w-6=0\qquad\Rightarrow w^2-4w-8=0$$

$$\Rightarrow w=\frac{4\pm\sqrt{48}}{2}=2\pm 2\sqrt{3}.$$

Case 1:

$$w=2+2\sqrt{3}\qquad\Rightarrow t+\frac{1}{t}=2+2\sqrt{3}$$

$$\Rightarrow t^2-\left(2+2\sqrt{3}\right)t+1=0$$

$$\Rightarrow t=\frac{2+2\sqrt{3}\pm\sqrt{12+8\sqrt{3}}}{2}\qquad=1+\sqrt{3}\pm\sqrt{3+2\sqrt{3}}$$

$$\Rightarrow \tan\left(\frac{x}{2}\right)=1+\sqrt{3}\pm\sqrt{3+2\sqrt{3}}$$

$$\Rightarrow \frac{x}{2}=\tan^{-1}\left(1+\sqrt{3}\pm\sqrt{3+2\sqrt{3}}\right)+n\pi,\quad n\in Z$$

$$\Rightarrow x=2\tan^{-1}\left(1+\sqrt{3}\pm\sqrt{3+2\sqrt{3}}\right)+2n\pi.$$

Case 2:

$w = 2 - 2\sqrt{3} \qquad \Rightarrow t + \frac{1}{t} = 2 - 2\sqrt{3}$

$$\Rightarrow t^2 - \left(2 - 2\sqrt{3}\right)t + 1 = 0$$

$$\Rightarrow t = \frac{2 - 2\sqrt{3} \pm \sqrt{12 - 8\sqrt{3}}}{2}.$$

$12 - 8\sqrt{3} < 0$ therefore $t = \frac{2 - 2\sqrt{3} \pm \sqrt{12 - 8\sqrt{3}}}{2}$ has no real solution and is rejected.