- The direction that the task takes is allowed to develop in a natural way.

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FIBONACCI AND FRACTIONS

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In 1202, Fibonacci introduced Hindu-Arabic arithmetic to Europe through his work Liber Abaci. Fibonacci presents an unusual approach to fractions. Why did he take this approach? Can we learn something about teaching students about fractions from Fibonacci?

Introduction

Leonardo Pisano (1170-1250), better known today by the name Fibonacci, contributed to the dissemination from East to West of the Hindu numerals and decimal place system and related arithmetic, through his seminal work *Liber Abaci* (or Book of Calculation). In this paper, we examine Fibonacci's treatment of fractions – specifically his notational system; and the rationale for, and applications of, the system.

It is generally acknowledged that fractions are amongst the most cognitively complex concepts taught in the primary mathematics curriculum (Cramer, Post, & delMas, 2002; Lamon, 2007; Mack, 1993; Smith, 2002; Streefland, 1991). One of the major factors that contribute to the difficulties associated with fractions are the many interpretations (or constructs) (Behr et al., 1993) that relate to rational numbers (Clarke, Sukenik, Roche, & Mitchell, 2006). Agreement exists that fractions can be interpreted as (a) a part to whole comparison, (b) a decimal, (c) a ratio, (d) a division, (e) an operator, and (f) a measure. The part-whole interpretation of rational numbers is a common focus of fraction instruction, and it is argued that emphasis on this interpretation can limit students' understanding of fractions (Lamon, 2007) because it can obscure the numerical nature of fractions implicit in the ratio, quotient, measure, and operator constructs (Domoney, 2002).

It is therefore of interest to investigate the history of the development of fraction notation and applications, and in particular, how Fibonacci presented these in what essentially is the first mathematics textbook of the Middle Ages. As Lenard, Itter, Mills & Yaneff (2007; p. 151) note, "*Liber Abaci*, acknowledged as a fundamental text in the history of mathematics, can make interesting and important contributions to a modern day mathematics classroom". It is with this objective in mind that we investigate Fibonacci's approach to fractions.

Fibonacci's notations for fractions

In Chapter 5 of *Liber Abaci,* Fibonacci introduces four different notations for fractions in addition to our usual notation. The first notation is illustrated by this example:

$$\frac{1\quad 2\quad 3\quad 4}{5\quad 6\quad 7\quad 8}\;9$$

$$= 9 + (4/8) + 3/(7\times 8) + 2/(6\times 7\times 8) + 1/(5\times 6\times 7\times 8).$$

Note that the integer part of the number, 9, appears on the right hand side of the number.

The second notation is illustrated as follows:

$$\frac{1\quad 2\quad 3\quad 4}{5\quad 6\quad 7\quad 8}\;\circ\;\;9$$

$$= 9 + (1\times 2\times 3\times 4)/(5\times 6\times 7\times 8) + (2\times 3\times 4)/(6\times 7\times 8) + (3\times 4)/(7\times 8) + 4/8.$$

The third notation is illustrated by:

$$\circ\;\frac{1\quad 2\quad 3\quad 4}{5\quad 6\quad 7\quad 8}\;9$$

$$= 9 + (1\times 2\times 3\times 4)/(5\times 6\times 7\times 8).$$

Fibonacci's fourth notation for fractions is illustrated by:

$$\frac{\underline{1}\quad \underline{2}\quad \underline{3}\quad 4}{5\quad 6\quad 7\quad 8}\;9$$

$$= 9 + 1/(5\times 8) + 2/(6\times 8) + 3/(7\times 8) + 4/8.$$

What was the purpose of these four notations? After reading *Liber Abaci*, one is impressed with the depth of Fibonacci's thinking about mathematics and computation. Thus, it would be rash to discard these notations as ridiculous. Fibonacci makes considerable use of the first notation; he makes a little use of the third notation (Sigler 2003, p. 439); the other notations are rarely used. In the next sectio, we offer an explanation as to how Fibonacci used the first notation. For historical aspects of the notation see Cajori (1974) and Djebbar (1992).

Applications of Fibonacci's notation

In this section we present several applications of Fibonacci's first notation. These will assist us to understand why he used it.

In writing *Liber Abaci,* Fibonacci had an eye on mercantile applications of mathematics. Consider the following application in finance. How would you express 12 pounds, 5 shillings, 7 pence, and 3 farthings, or £12/5/7¾ , in pounds? Recall that there are 20 shillings in a pound, 12 pence in a shilling, and 4 farthings in a penny. To express this in pounds using decimal fractions is difficult. However, in Fibonacci's notation, it is easy: £12/5/7¾ can be written as:

$$\pounds\;\frac{3\quad 7\quad 5}{4\quad 12\quad 20}\;12.$$

Here is a more contemporary application. Express 3 weeks, 2 days, 11 hours, 12 minutes, and 21 seconds in weeks. For our students in 2008, this would be a challenging calculation. Eight centuries ago, a student of Fibonacci could write down the answer immediately as:

$$\frac{21\quad 12\quad 11\quad 2}{60\quad 60\quad 24\quad 7}\;3.$$

To appreciate Fibonacci's notation from a more abstract point of view, consider the relationship between our decimal notation and Fibonacci's notation for fractions. In our notation

$$1.2345 = 1 + 2/(10) + 3/(10\times 10) + 4/(10\times 10\times 10) + 5/(10\times 10\times 10\times 10)$$

and, in Fibonacci's notation, 1.2345 is:

$$\frac{5\quad 4\quad 3\quad 2}{10\quad 10\quad 10\quad 10}\;1.$$

Fibonacci's notation is more general because it can deal with expressing fractions in bases other than base 10.

Fibonacci's notation and division

In this section we examine how Fibonacci's notation is used in division. Suppose that one wanted to calculate 112358 ÷ 263. Note that 263 = 3×7×13. Now perform the division as follows.

$$112358 \div 263 \qquad \text{(The problem)}$$

$$= 112358 \times \frac{1\quad 0\quad 0}{3\quad 7\quad 13} \qquad \text{(Write 1/263 in Fibonacci's notation)}$$

$$= \frac{2}{3}\ 37452 \times \frac{1\quad 0}{7\quad 13} \qquad \text{(Divide by 3)}$$

$$= \frac{2\quad 2}{3\quad 7}\ 5350 \times \frac{1}{13} \qquad \text{(Divide by 7)}$$

$$= \frac{2\quad 2\quad 7}{3\quad 7\quad 13}\ 411 \qquad \text{(Divide by 13)}$$

Note that Fibonacci expects his readers to know their 13-times tables.

However, it is in checking the calculation that we can really see the power of this notation. To check the above division, we want to know:

Does 112358 = 411×(3×7×13) + 7×(3×7) + 2×(3) + 2 ?

Does 112358 = (((411×13 + 7)×7 + 2)×3 + 2) ?

Suppose that we check to see if these are equal by casting out 5's. Note that 112358 ≡ 3 mod 5. Fibonacci's notation helps us to evaluate the right hand side, modulo 5, easily: start with 411 ≡ 1 (mod 5); 1×13 ≡ 3 (mod 5); 3+7 ≡ 0 (mod 5); 0×7 ≡ 0 (mod 5); 0+2 ≡ 2 (mod 5); 2×3 ≡ 1 (mod 5); and finally 1+2 ≡ 3 (mod 5). Thus, the left hand side and the right hand side are equal modulo 5.

Furthermore, the observation that 411×(3×7×13) + 7×(3×7) + 2×(3) + 2 is equivalent to (((411×13 + 7)×7 + 2)×3 + 2) reminds us of the observation in contemporary numerical mathematics that the second form is less complex than the former because it involves fewer multiplications. We can see the elements of modern complexity theory in *Liber Abaci*.

Conclusion

Fibonacci's treatment of division explicitly acknowledges the relationship between division, and multiplication by the inverse of the divisor. It also reiterates a method of approaching a division problem involving a large composite divisor by sequentially dividing by the factors of that divisor. His method deals effectively with the remainders resulting from each step of the division, and the answer could easily be manipulated to be represented as a whole number and a single fraction. In addition, Fibonacci's introduction and elaboration of fractions is articulated within the context of division, the only operation with whole numbers that will produce fractions. A division context provides a useful site to explore fractions in a meaningful way, emphasising their connection with other concepts.

It is common practice in schools to present fractions as isolated phenomena, with particular emphasis on the part-whole interpretation (Lamon, 2007) or construct (Behr, et al., 1993). Exclusive focus on the part-whole construct can limit students' understanding of the different interpretation of fractions, particularly that of a quotient (or implied division). Fibonacci briefly introduces his fraction notation, and then immediately moves to a division context, emphasising the quotient interpretation of fractions. He does not deal explicitly with the part-whole interpretation at all. This in itself is an interesting feature of his treatment of fractions.

This study of one aspect of the history of fractions illustrates how the history of mathematics alerts us to things forgotten over time. We can develop a deeper understanding of our subject by studying classic texts such as *Liber Abaci*, or Euclid's *Elements*. On the history of mathematical notation, Cajori (1974) is particularly fascinating.

If today's students do not like fractions, they ought to be grateful that Fibonacci's four notations have not survived the passing of time.

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SWITCHING OFF MATHEMATICS: VALUES THAT INHIBIT LEARNING

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Values have been a relatively recent focus in mathematics education research. However it requires further attention in the mathematics classroom. If teachers could focus on what makes students 'switch on or off' in learning mathematics, they would be able to sculpture a desired social setting and complement it with an effective pedagogy to maintain excitement and promote effective mathematics learning. An interview to help us determine the values that inhibit learning could lead us to understand what does and does not work for students in learning mathematics. It is therefore reasonable and possible to examine values from a different point of view: initially a negative one, which will then steer into narrowing the positive values in mathematics education that teachers could/should embrace.

Australia's performance in PISA

The PISA exam measures how well young adults (age 15) are able to meet the challenges of today's society; their ability to use their knowledge and skills to meet real life challenges (OECD (2004), pg 20. The survey administered covers mathematics (the main focus in 2003), reading, science and problem solving. PISA considers student knowledge in these areas not in isolation but in relation to students' ability to reflect on their knowledge and experience and to apply them to real world issues.