With best wishes,
Jamie Simpson

CROSS SECTION

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We convert fractions to decimals by simple division. Here are some examples.

$$\frac{1}{4} = 0.25 \text{ and } \frac{2}{5} = 0.4.$$

Sometimes, we get repeating decimals that go on forever:

$$\frac{1}{3} = 0.333... = 0.\overline{3} \text{ and } \frac{4}{99} = 0.\overline{040404} = 0.\overline{04}.$$

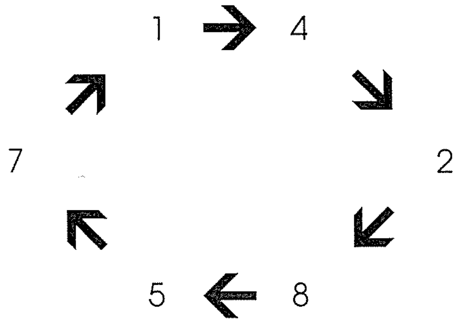
Strictly speaking, 0.25 is a repeating decimal too because 0.25 = 0.25000... .

Pagni (2004) argues that teachers should stress that fractions are repeating decimals and vice-versa. Beswick (2004) explains why we regard 0.999... and 1.000... as equal.

An interesting pattern emerges when we express 7-ths as decimals. Write 1/7 as:

$$\frac{1}{7} = 0.142857142857... = 0.\overline{142857}.$$

We can represent this infinite decimal in a circle.



To read off the decimal expansion of 1/7, we start at 1 and work around the circle clockwise over and over again without stopping and obtain the infinite decimal expansion of 1/7.

Now here is a curious fact. To read off the decimal expansion of 2/7, we start at 2 and work around the same circle clockwise. Thus, 2/7 = $0.\overline{285714}$. If you don't believe this, check it out.

To write 3/7 as a decimal, we start at 4 and work around the circle clockwise. And so on. To write 6/7 as a decimal, we start at 8 and work around the circle clockwise.

The fractions 1/7, 2/7, ..., 6/7 lead to some curious decimals.

Here is an exercise. Discover a similar circular phenomenon with the fractions 1/17, 2/17, ..., 16/17. Likewise, check 1/19, 2/19, ... , 18/19. The answers are at the end of this article.

If you can find such circular patterns in the decimal expansion of *1/p* for a prime number *p*, then *p* is called a "long prime". Thus, 7, 17, and 19 are three long primes. Obviously 5 is not a long prime because the decimal expansions of 1/5, 2/5, 3/5, 4/5 do not fit in with this circular pattern.

Why are such primes described as "long"?

If *p* is a prime number, then 1/*p* can be written as an infinite repeating decimal like this:

$$\frac{1}{7} = 0.\overline{r_1 r_2 r_3 \ldots r_k}.$$

It turns out that *p* if is a prime number then $k < p$. The largest possible value of k is $p-1$, or, the maximum length of the repeating block of digits is $p-1$. The proof of these facts is not so simple; see Hardy and Wright (1960, Chapter 9). If *p* is a prime, and in the decimal expansion of 1/*p* we have a repeating block of length $p-1$, then we say that *p* is a "long prime".

When $p=7$ we have $k=6 = p-1$ so 7 is a long prime.

When $p=5$, $1/p = 0.2000...$ we have $k=1$, which is not $p-1$, and therefore 5 is not a long prime.

When $p=11$, $1/11 = 0.090909...$, we have $k=2$, which is not $p-1$, and so *11* is not a long prime either. You can check that 17 is a long prime.

If *p* is a long prime, then the decimal expansion of 1/*p* will exhibit the circular pattern seen above for 1/7. You will be able to read off the decimal expansions of $1/p$, $2/p, \ldots, \cancel{(1/p)}/p$ by travelling around the same circle.
(p−1)/p

There are infinitely many prime numbers. Are there infinitely many long primes? Nobody knows the answer to this question, although mathematicians suspect that there are infinitely many long primes (Conway and Guy, 1996, p. 170).

Starting with a simple idea of turning fractions into decimals, all we have to do is step off the track and we arrive at the frontiers of knowledge! This situation is not uncommon in mathematics. Melrose and Scott (2005) describe how some simple questions about prime numbers can be fiendishly difficult to answer.

There is another interesting property of sevenths. We know that $1/7 + 6/7 = 1$, so we could think of 1/7 and 6/7 as twins. We know that $1/7 = 0.\overline{142857}$. The first 3 digits in the cycle are 142 and the second 3 digits are 857. Using the circle, we find that $6/7 = 0.\overline{857142}$. Here the first 3 digits are 857 and the second 3 digits are 142. In other words, the first half of the cycle for 1/7 and the second half are reversed for its twin 6/7.

We detect similar patterns when we compare $2/7 = 0.\overline{285714}$ with $5/7 = 0.\overline{714285}$ or $3/7 = 0.\overline{428571}$ with $4/7 = 0.\overline{571428}$.

You would also find this effect in the long primes 17 and 19 (and, in fact, for all long primes). Kaplan (n.d.) describes this effect and other interesting properties in his online article.

In our experience, few citizens know about the pattern associated with the decimal expansions of 7-ths. This is a curious, even beautiful, pattern. It leads into discussion of long division. Students become aware of the limitations of the hand-held calculator or a spreadsheet. The topic may elicit questions and numbers and encourage students to explore number patterns.

Some answers

The numbers 17, 19, and 23 are long primes as shown by the following decimal expansions. You may find that your calculator does not show sufficient decimal places to be useful for these long primes. Even a spreadsheet may not give you a sufficient number of decimal places to confirm that a prime is a long prime. You may need to resort to long division—as we did!

$1/7 = 0.\overline{05882352294117647}$. Note that there are 16 = 17-1 digits in the cycle and the twin effect involves two blocks of 8 digits.

$1/19 = 0.\overline{052631578947368421}$. There are 18 = 19-1 digits in the cycle and the twin effect involves two blocks of 9 digits.

$1/23 = 0.\overline{0434782260869565217391 3}$. There are 22 = 23-1 digits in the cycle and the twin effect involves two blocks of 11 digits.

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