BEYOND THE RATIONAL

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Rational and irrational numbers arise in Year 10A of the Australian Curriculum. What should students learn about irrational numbers? Why should students learn about irrational numbers? What interesting exercises about irrational numbers are suitable for Year 10A? How can we connect irrational numbers to other parts of mathematics, or other fields of study? We consider these questions in light of the aims of the Australian Curriculum. Also, we explain why irrational numbers are fascinating.

Introduction

According to the Australian Curriculum, students in Year 10A are expected to be able to "define rational and irrational numbers and perform operations with surds and fractional indices". (All quotes in this paper are from the Australian Curriculum v. 7.4 at http://www.australiancurriculum.edu.au/.)

Let us recall the definitions of rational and irrational numbers (Garner et al. (2012, p. 26).

A rational number is a number that can be expressed as m/n where m,n are integers and $n \neq 0$. An irrational number is a real number that cannot be expressed as the ratio of two integers.

In this paper we will address the following questions. What should students learn about irrational numbers? Why should students learn about irrational numbers? What interesting exercises about irrational numbers are suitable for Year 10A? How can we connect irrational numbers to other parts of mathematics, or other fields of study? We consider these questions in light of the aims of the Australian Curriculum. Finally, we will share our fascination with irrational numbers.

What Should Students Learn about Irrationals?

Definitions are important in life, and mathematics is one subject where good use is made of definitions. The above quote from the Australian Curriculum makes it clear that students ought to be able to define rational and irrational numbers. It is worthwhile to emphasise that irrational numbers are defined as numbers that do not have a particular property: they cannot be expressed as a ratio of two integers.

Do Irrational Numbers Exist?

Irrational numbers do exist. Since the proof that $\sqrt{2}$ is irrational is well known, we present a different example. Consider $\log_{10}(3)$ (the logarithm of 3 to the base 10). This number is positive. If it is rational, then there exist positive integers m,n such that $\log_{10}(3) = m/n$, and so $10^{(m/n)} = 3$, or $10^m = 3^n$. This is impossible because 10^m is an even number (when $m > 0$) and 3^n is an odd number. Therefore $\log_{10}(3)$ cannot be rational; it must be irrational. This cute argument is suitable for Year 10A students once they have encountered logarithms.

In general, it is not straightforward to show that a particular number is irrational. You cannot do it on a calculator. For example, we know that π and e are irrational – but proving these facts is beyond Year 10A. And, surprisingly, nobody knows whether $\pi + e$ is rational or irrational! Mathematicians have proved that e^{π} is irrational, but nobody knows whether π^e is rational or irrational.

Once irrational numbers have been defined, it is reasonable to expect that students know that irrational numbers exist. Someone should prove to them that irrational numbers exist. Perhaps they can be convinced by the argument above about $\log_{10}(3)$ or the classical arguments about $\sqrt{2}$. However, asking students whether a given number is rational or irrational is not suitable for Year 10A students. If the number is rational then the question is trivial; if the number is irrational then students cannot be expected to understand why the number is irrational except in a few special cases.

Surds

A "surd" is usually defined as an irrational number that is a root of a rational number; see for example Garner et al. (2012, p. 26). The ability to "perform operations with surds and fractional indices" is important, but it is part of arithmetic and algebra, and not connected with irrational numbers *per se*.

Why Should Students Learn about Irrationals?

Irrational numbers do not seem to be useful in any application that we might regard as practical. Even the irrationality of π is not relevant to its many applications. There is a challenge in demonstrating to students that irrational numbers are relevant to "their personal, work and civic life".

The Number Line

On the other hand, the concepts of rational and irrational numbers illustrate the beauty and complexity of the simple number line.

The number line, sometimes called the real line, contains all the real numbers. Every point on the line represents exactly one real number, and every real number is represented by exactly one point on the line. At first sight, the real line seems to be a rather boring object.

Between any two rational numbers on the real line, you will find another - just take the average of them. Indeed, any interval on the real line will contain infinitely many rational numbers. Rational numbers are almost everywhere on the real line.

Suppose that you remove the point representing the integer 7 from the number line. What do you see? Although there is a gap at 7, you would not notice it because it is infinitely small. Now remove another point, say the one corresponding to the integer 5. Still, you cannot notice the difference. In fact, if you remove all the points corresponding to integers you will not notice any difference in the real line.

Now here is an astonishing fact. Even if you remove all the points corresponding to rational numbers, you will not notice the difference!

We have an amazing state of affairs. On one hand, every interval on the real line, no matter how tiny, contains infinitely many rational numbers. On the other hand, if you remove all the rational numbers from the real line, you do not notice any gaps in the line. Almost every real number is irrational. If you remove all the points corresponding to irrational numbers, the line would disintegrate! Here we have been using ideas about measure, topology and infinity informally. Stillwell (2010, Chapter 1) provides a more formal description of the situation.

The concepts of rational and irrational numbers make the real line more complicated – and more interesting – than it first seems. They illustrate that "mathematics has its own value and beauty".

Even a brief reference to these ideas in the classroom contributes to realising one of the aims of the Australian Curriculum, namely "to ensure that students ... develop an increasingly sophisticated understanding of mathematical concepts".

Exercises on Irrationals

The study of irrational numbers cannot be conducted to any substantial depth in Year 10A: the subject gets too hard too quickly. Even the question "Is π rational or irrational?" is not suitable for Year 10A students because they cannot be expected to be able to explain or understand why this number is irrational. The challenge is to find exercises on irrational numbers that are mathematically sound, and suitable for Year 10A students.

It has been often said that proof is the glue that holds mathematics together. We can use irrational numbers to introduce students to this essential part of mathematics. Here we suggest some exercises about irrational numbers that may be suitable for the Year 10A classroom. Most of these problems involve very simple proofs. A teacher might show the students how to solve these problems and then let them reproduce it. The satisfaction comes from seeing, comprehending, sharing and reproducing a short proof in its entirety.

- Explain why the following numbers are rational numbers:
  $\frac{22}{7}, 4, -\frac{5}{10}, \frac{2\sqrt{3}}{7\sqrt{3}}, \frac{(\sqrt{3}+1)(\sqrt{3}-1)}{4}, 1.21, 1.\dot{2}, 1.\dot{2}\dot{1}.$
- Prove that the sum of two rational numbers is a rational number. [Hint: Show that $2/3 + 4/5$ is rational; then generalise to $m/n + p/q$.]
- Prove that it is possible for the sum of two irrational numbers to be a rational number. [Hint: You may use the fact that $\sqrt{2}$ is irrational.]
- In the previous two problems, replace "sum" by "product". [Hint: You may use the fact that $\sqrt{2}$ is irrational.]
- Prove that the average of two rational numbers is a rational number.
- You have just proved that the average of two rational numbers must be a rational number. Use this repeatedly to prove that, between any two given rational numbers, there are infinitely many rational numbers.
- Prove that it is possible for the average of two irrational numbers to be a rational number. [Hint: You may use the fact that $\sqrt{2}$ is irrational.]
- If we assume that π is an irrational number, prove that $\pi + 5$ is also an irrational number.
- The number $\varphi = (1 + \sqrt{5})/2$ is a famous irrational number, sometimes called the golden ratio. Prove that $\varphi^2 = \varphi + 1$. Then use this to show that $\varphi^3 = 2\varphi + 1$ and $\varphi^4 = 3\varphi + 2$.
- Prove that $\varphi = 1 + \frac{1}{1+\frac{1}{\varphi}}$. This, and many other identities, can be found in Posamentier & Lehmann (2012).

- A calculator says that $\sqrt{2} = 1.4142136$. Prove that this is not correct by calculating the exact value of $(1.4142136)^2$.
- Prove that $\log_{10}(5)$ is irrational. Prove that $\log_{10}(2)$ is irrational. Then observe that $\log_{10}5 + \log_{10}2 = 1$ which is not rational.
- Write a short letter to a Year 9 student in which you explain the difference between rational and irrational numbers.

This is a short list – but irrational numbers form only a small part of Year 10A.

Collectively, these problems cover several general capabilities recommended for Year 10A in the Australian Curriculum such as "comprehend texts", "understand learning area vocabulary", "compose texts", "identifying, exploring and organising information and ideas" and "develop ICT capability". The problems also offer opportunities for students to "work collaboratively in teams" which is a personal and social capability listed for the curriculum. Giving students in Year 10A a little experience in reproducing "previously seen simple mathematical proofs" prepares them for more advanced work in Specialist Mathematics.

Irrationals and Other Topics

One of the aims of the Australian Curriculum is to ensure that students "recognise connections between the areas of mathematics and other disciplines". How can we connect the study of rational and irrational numbers to other topics?

We have shown how knowledge of rational and irrational numbers increases our understanding of the number line.

In the language of the Australian Curriculum, rational numbers have decimal expansions that are either "terminating" or "recurring and non-terminating". Irrational numbers have decimal expansions that are non-terminating and non-recurring. Consideration of irrational numbers in Year 10A deepens the understanding of decimal representations encountered in Year 8.

The exercises listed above demonstrate the power of algebra.

The example about $\log_{10}(3)$, and some of the exercises above, link irrational numbers to logarithms, and indices both of which are encountered in Year 10A.

Thinking about rational and irrational numbers leads to thinking about infinity. Although the Year 10A classroom is not the place to discuss infinity, a discussion of rational and irrational numbers is on the verge of touching on this subject that has intrigued human beings for thousands of years. Stillwell (2010), and, to a lesser extent, Rucker (1982) discuss the association between rational and irrational numbers, and infinitely small and infinitely large numbers.

In her biography of Pythagoras, Ferguson (2010), explains why the discovery of irrational numbers was so shocking; Fritz (1946) presents a mathematical, but readable, account of the discovery of irrationals; Havil (2012) describes the history of irrational numbers from ancient Greece to modern times. Studying irrational numbers affirms that mathematics "has its origin in many cultures".

As shown in one of the exercises above, the golden ratio $\varphi = (1 + \sqrt{5})/2$ is a solution a quadratic equation – another topic in Year 10A. It also has connections with many fields quite different from mathematics; see Rossi (2004) and Posamentier & Lehmann (2012).

Advances in knowledge often stem from making connections between ideas. The topic of irrational numbers allows students to make such connections. Connecting ideas brings strength and cohesion to our understanding of the world.

The Fascination of Irrationals

In this section, we share our fascination with irrational numbers and stray from our emphasis on the Year 10A classroom.

Irrational numbers play a deep role in other branches of mathematics. A number of results in mathematics are based on the irrationality of some certain numbers. Irrational numbers are a main part of the system of real numbers, which is the foundation of analysis. The proof of irrationality of particular numbers usually requires deep methods from analysis, and hence, provides an impetus for advances in mathematical analysis.

While irrational numbers have almost no application in practice, they provide examples of numbers that have no pattern in the digits in their decimal representation. This motivates computer scientists to attack the problem of computing their digits to a large number of decimal places. Such problems provide an impetus for advances in computer science. For other applications in information sciences see Schroeder (1997).

Conclusions

Rational and irrational numbers form a small topic in Year 10A of the Australian Curriculum in Mathematics. Discussion of these numbers can advance several aims and aspirations of the Australian Curriculum. By considering one topic, one gains a deeper appreciation of the Australian Curriculum.

We conclude with an exercise. Let $z = x^y$. There are four possibilities: x might be rational or irrational; y might be rational or irrational. Find examples that show that, in each of the four cases, z can be rational or irrational. [Hint: There are many solutions. We used only the facts that $\sqrt{2}$ is irrational, $\sqrt{3}$ is irrational, $\log_2 3$ is irrational, and the laws of indices and logarithms. See also Jones & Toporowski (1973).]

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References

Ferguson, K. (2010). *Pythagoras: His lives and the legacy of a rational universe.* Crows Nest: Allen & Unwin.

Fritz, K. von. (1946). The discovery of incommensurability by Hippasus of Metapontum. *Annals of Mathematics, 46(2)*, 242-264.

Garner, S., Brodie, R., Corcoran, S., & Swift, S. (2012). *Nelson think maths for the Australian Curriculum 10 + 10A*. South Melbourne: Cengage.

Hardy, G. & Wright, E. (2008). *An introduction to the theory of numbers.* Sixth ed. Oxford: Oxford University Press.

Havil, J. (2012). *The irrationals: A story of the numbers you can't count on.* Princeton: Princeton University Press.

Jones, J.P. & Toporowski, S. (1973). Irrational numbers. *Amer. Math. Monthly, 80*(4), 423-424.

Niven, I. (1961). *Numbers: Rational and irrational.* Second ed. New York: Random House.

Posamentier, A.S. & Lehmann, I. (2012). *The glorious golden ratio.* Amherst: Prometheous Books.

Rossi, C. (2004). *Architecture and mathematics in ancient Egypt.* Cambridge: Cambridge University Press.

Rucker, R. (1982). *Infinity and the mind: The science and philosophy of the infinite.* Boston: Birhäuser.

Schroeder, M. (1997). *Number theory in science and communication.* Third ed., Berlin: Springer.

Stillwell, J. (2010). *Roads to infinity: The mathematics of truth and proof.* Natick: A K Peters.