SUMMER QUIZ 2012
SOLUTIONS – PART 3

Hard 1

$$2 = \sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\ldots}}}}}}}}$$

Solve for *x*.

Answer: $x = 2$.

Squaring both sides, we see

$$4 = x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\cdots}}}}}}} = x \times 2$$

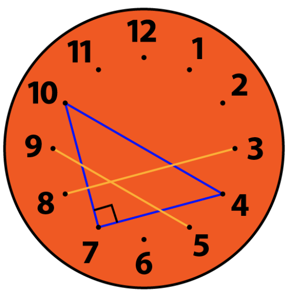
It follows that $x = 2$.

Hard 2

You draw a line connecting the 5 and 9 on a clock face, and another line connecting the 3 and 8. What is the angle between the two lines?

Answer: 45 degrees.

The two lines are parallel to the two short sides of the highlighted right-angled triangle. This triangle is clearly half of a square, and so the angle in question is 45 degrees.



Hard 3

In his will, a man leaves 7 full, 7 half-full and 7 empty wine barrels to his three sons. Every son is supposed to receive the same number of barrels and the same amount of wine. How can this be done? You are not allowed to pour wine from one barrel to another.

Answer: Each son is supposed to receive 7 barrels and $(7 + 7/2)/3 = 3.5$ barrels of wine. So, each son must receive at least one half-full barrel, plus a further three barrels of wine. There are then two ways of distributing the wine:

Son 1: (1 half-full barrel) + (3 full barrels)

Son 2: (1 half-full barrel) + (2 full barrels + 2 half-full barrels)

Son 3: (1 half-full barrel) + (2 full barrels + 2 half-full barrels)

Or

Son 1: (1 half-full barrel) + (3 full barrels)

Son 2: (1 half-full barrel) + (3 full barrels)

Son 3: (1 half-full barrel) + (1 full barrel + 4 half-full barrels)

It is then easy to complete these two solutions by distributing the empty barrels:

Son 1: (1 half-full barrel) + (3 full barrels) + (3 empty barrels)

Son 2: (1 half-full barrel) + (2 full barrels + 2 half-full barrels) + (2 empty barrels)

Son 3: (1 half-full barrel) + (2 full barrels + 2 half-full barrels) + (2 empty barrels)

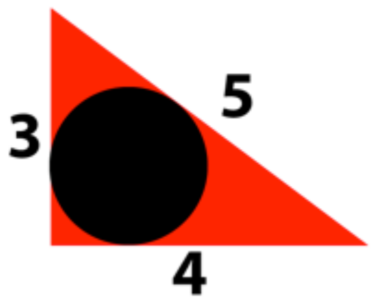
Or

Son 1: (1 half-full barrel) + (3 full barrels) + (3 empty barrels)

Son 2: (1 half-full barrel) + (3 full barrels) + (3 empty barrels)

Son 3: (1 half-full barrel) + (1 full barrel + 4 half-full barrels) + (1 empty barrel)

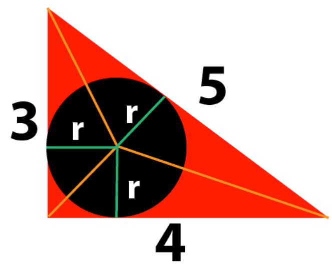
Hard 4



What is the radius of the circle?

Answer: 1.

The area of the triangle is obviously 1/2 x 3 x 4 = 6. We can also calculate the area in a non-obvious way, by drawing altitudes from the centre of the circle:



Summing the areas of all the sub-triangles, we obtain 1/2 x (3 + 4 + 5) x *r*. Equating the two areas, we see *r* = 1.

Hard 5

A postman knocks on the door of a house where two children live, and the door is answered by a little girl. What are the chances that the other child is also a girl?

Answer: 1/3.

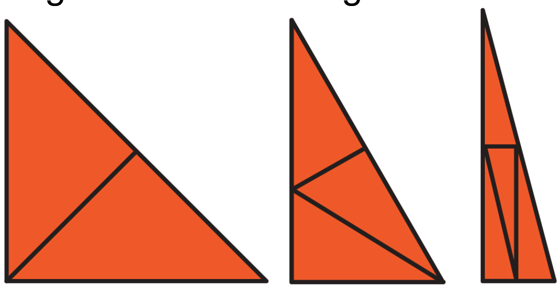
Before the postman knocks, there are four possibilities: both children are boys; both children are girls; the younger child is a boy and the older a girl; the younger child is a girl and the older a boy. Once the door is answered by a girl, there are only three possibilities left, and in exactly one of these is the other child a girl. So, the chances that the second child is a girl is 1/3.

Hard 6

The right-angled triangles that make up the façade of Federation Square have the beautiful property that five of the triangles can be combined into a larger triangle of the same shape. Can you find a right-angled triangle for which two such triangles combine into a triangle of the same shape? What if you want three triangles to combine into the same shape, or four?

Answer: Yes.

It is easy to see that two copies of a 45-45-90 right-angled triangle combine into a larger 45-45-90 triangle. More surprising is that three copies of a 30-60-90 triangle combine into a larger 30-60-90 triangle.



Combining four triangles is much easier, and can be done with any shape right-angled triangle. Combining more than five triangles is much trickier, and cannot always be done. These are called rep-tiles, and there are many good articles to be found on them:

http://www.uwgb.edu/dutchs/symmetry/reptile1.htm

Hard 7

You have nine square tiles: three red, three blue and three yellow. In how many ways can you arrange the tiles into a 3 x 3 square so that no two tiles of the same colour meet along an edge?

Answer: 36.

This requires some careful counting. To begin, we'll assume the top left square is red and the top middle (1,2) square is yellow: label these squares as (1,1) and (1,2) respectively. We now consider two cases; Case 1, where the (2,1) square is yellow; and Case 2, where the (2,1) square is blue.

CASE 1: (1,1) is red, (1,2) is yellow and (2,1) is yellow (as pictured above)

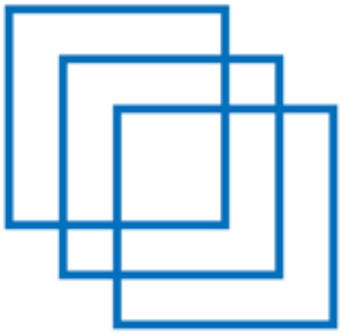
If the (2,2) square is red, then the blue squares must be (1,3), (3,1) and (3,3), which is impossible: there is no space left for the third red square. If the (2,2) square is blue, then the other blue squares must two of (1,3), (3,1) and (3,3). This gives three configurations, all of which are possible to complete, and each in only one way.

CASE 2: (1,1) is red, (1,2) is yellow and (2,1) is blue.

In this case (2,2) must be red. The remaining yellow squares must be either: (3,1) and (2,3); or (3,1) and (3,3); or (3,2) and (2,3). It can be checked that all these configurations can be completed, and in only one way.

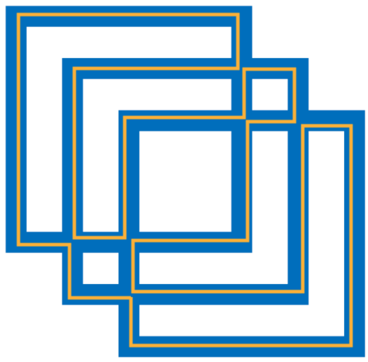
In total, we have found 6 tilings. However, we had three choices for the colour at (1,1), and then 2 for the choice of colour at (1,2). That means the total number of possible tilings is 6 x 3 x 2 = 36.

Hard 8



Draw this combination of squares as one path. The path cannot cross itself, and you cannot lift your pen from the paper.

Answer:



Hard 9

$$a \times b \times c = a + b + c$$

You are told to multiply three numbers together but instead you add them. By luck, you wind up with the correct answer. What are the numbers?

Answer: 3, 2 and 1 (assuming all are positive whole numbers).

We'll assume a, b and c are all positive whole numbers. (Otherwise, there are lots of solutions as long as $b \times c$ doesn't equal 1.) By switching the numbers around, we can also assume $a \geq b \geq c$. Then

$$3a \geq a + b + c = abc$$

That means (since we're assuming a cannot be 0) that $3 \geq bc$.

Since we're assuming b and c are positive whole numbers, that leaves three possibilities: $b = 1$ and $c=1$; $b = 2$ and $c = 1$; $b = 3$ and $c = 1$. Checking all three possibilities we find that the only solution is $a = 3$, $b = 2$, $c = 1$.

Hard 10

An enclosure contains 12 blue, 16 green and 20 red chameleons. Whenever two chameleons of different colours meet they both change to the third color. Can all the chameleons wind up the same colour?

Answer: No.

We'll let B, G and R stand for the number of blue, green and red chameleons at any time. Now consider what can happen to the difference $G - B$ when two chameleons of different colours meet.

If green and blue meet then both become red and $G - B$ remains the same.

If green and red meet then we have one less green and two more blues, and so $G - B$ goes down by 3.

If red and blue meet then, calculating as above, $G - B$ goes up by 3.

It follows that any meeting results in $G - B$ changing by a multiple of 3. Since we started with $G - B = 4$, we can never wind up with $G - B = 0$, and in particular we couldn't wind up with only red chameleons (i.e. $G = B = 0$). Since we also start with $R - B = 8$ and $R - G = 4$, the exact same argument works for these differences as well.