

For example, if $g(x) = x^{10}$, then

$$f(x) = \frac{1}{2} \int \left(x^{10} - \frac{1}{x^{10}} \right) dx = \frac{x^{11}}{22} + \frac{1}{18x^9} (+C).$$

Or try $g(x) = \tan x$. Then the indefinite integral for $f(x)$ can be computed, using some trig identities, as

$$\begin{aligned} \frac{1}{2} \int (\tan x - \cot x) dx &= \frac{1}{2} \left(-\ln \left(\frac{1}{2} \sin 2x \right) \right) + C \\ &= -\frac{1}{2} \ln(\sin 2x) - \frac{1}{2} \ln \left(\frac{1}{2} \right) + C. \end{aligned}$$

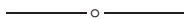
By ignoring the constants, we can choose $f(x) = \frac{1}{2} \ln(\sin 2x)$. However, although $f'(x)^2 + 1$ will not then equal $\left(\frac{1}{2} \left(g(x) + \frac{1}{g(x)} \right) \right)^2$, things still come out nicely:

$$\begin{aligned} \int \sqrt{(f'(x))^2 + 1} dx &= \int \sqrt{\cot^2 2x + 1} dx = \int \csc 2x dx \\ &= -\frac{1}{2} \ln |\csc 2x + \cot 2x| + C \quad (\text{for } 0 \leq x \leq \frac{\pi}{2}). \end{aligned}$$

We invite the reader to experiment with this algorithm and discover other examples.

References

1. C. H. Edwards and D. E. Penney, *Calculus*, 6th ed., Prentice Hall, 2002.
2. G. B. Thomas, Jr., R. L. Finney, M. D. Weir, and F. R. Giordano, *Thomas' Calculus*, 10th ed., Addison-Wesley, 2001.



Arc Length and Pythagorean Triples

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In this note we give an example of how a computer algebra system can offer surprises even in the context of a standard calculus topic. When introducing the formula for arc length, some natural examples are the curves C_n which are given parametrically by $x = t^n$, $y = t^{n+1}$, $0 \leq t \leq 1$, (n is a positive integer). Many students have difficulty computing even the length of C_1 by hand, so this is a natural place to use a computer algebra system. The length of C_5 , for example, is

$$\frac{3431\sqrt{61}}{20736} + \frac{15625}{124416} \ln 5 - \frac{15625}{124416} \ln(-6 + \sqrt{61}).$$

As n increases, the results become increasingly unpleasant until, surprisingly, we find that the length of C_{20} is rational and equals $\frac{36495661067145135829027}{25798674916142804999323}$.

It is easy to see why this is so and to show that infinitely many of the lengths $L(C_n)$ are rational numbers. Using the standard formula $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} dt$ for arc length, we have

$$L(C_n) = \int_0^1 t^{n-1} \sqrt{n^2 + (n+1)^2 t^2} dt.$$

After we make the successive substitutions $u = \frac{n+1}{n}t$ and $v = \sqrt{u^2 + 1}$, the integrand becomes $(v^2 - 1)^{(n-2)/2} v^2$.

We now take n to be even, say $n = 2k$. Then our integrand is just a finite sum of integer multiples of powers of v ; that is,

$$(v^2 - 1)^{(n-2)/2} v^2 = \sum_{j=1}^k c_{2j} v^{2j}.$$

for some integers c_2, c_4, \dots, c_{2k} . Integrating and substituting back, we find that

$$L(C_n) = \frac{n^{n+1}}{(n+1)^n} \sum_{j=1}^k \frac{c_{2j}}{2j+1} (u^2 + 1)^{(2j+1)/2} \Big|_0^{n+1},$$

and hence that

$$L(C_n) = \frac{n^{n+1}}{(n+1)^n} \sum_{j=1}^{n/2} \frac{c_{2j}}{2j+1} \left(\left(\frac{\sqrt{(n+1)^2 + n^2}}{n} \right)^{2j+1} - 1 \right).$$

This will be rational if $n^2 + (n+1)^2$ is a perfect square; that is, if n and $n+1$ are part of a Pythagorean triple. It is well known that there are infinitely many such pairs (see, for example [1, p. 164, Exer. 17]). In particular, if $(a, a+1, c)$ is a Pythagorean triple, so is $(3a+2c+1, 3a+2c+2, 4a+3c+2)$. Note that the parity of the first term switches, so that by using this result twice we can go from one even case to another. Consequently, not only does C_{20} have a rational length, so does C_{696} , and also infinitely many other curves C_n .

References

1. J. K. Strayer, *Elementary Number Theory*, PWS Publishing, 1994.



On the Convergence of Some Modified p -Series

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The series we consider, which we call (S, p) -series, are obtained from the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ by removing all those terms in which the base n in the denominator contains a digit in a specified set S . For example, if $S = \{1, 2\}$, then our series is

$$\frac{1}{3^p} + \frac{1}{4^p} + \dots + \frac{1}{9^p} + \frac{1}{30^p} + \frac{1}{33^p} + \frac{1}{34^p} + \dots + \frac{1}{40^p} + \frac{1}{43^p} + \frac{1}{44^p} + \dots$$