For example, if $g(x) = x^{10}$, then

$$f(x) = \frac{1}{2}\int\left(x^{10} - \frac{1}{x^{10}}\right)dx = \frac{x^{11}}{22} + \frac{1}{18x^9}\ (+C).$$

Or try $g(x) = \tan x$. Then the indefinite integral for $f(x)$ can be computed, using some trig identities, as

$$\frac{1}{2}\int(\tan x - \cot x)\,dx = \frac{1}{2}\left(-\ln\left(\frac{1}{2}\sin 2x\right)\right) + C$$
$$= -\frac{1}{2}\ln(\sin 2x) - \frac{1}{2}\ln\left(\frac{1}{2}\right) + C.$$

By ignoring the constants, we can choose $f(x) = \frac{1}{2}\ln(\sin 2x)$. However, although $f'(x)^2 + 1$ will not then equal $\left(\frac{1}{2}\left(g(x) + \frac{1}{g(x)}\right)\right)^2$, things still come out nicely:

$$\int\sqrt{(f'(x))^2 + 1}\,dx = \int\sqrt{\cot^2 2x + 1}\,dx = \int \csc 2x\,dx$$
$$= -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C \quad (\text{for } 0 \le x \le \tfrac{\pi}{2}).$$

We invite the reader to experiment with this algorithm and discover other examples.

References

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Arc Length and Pythagorean Triples

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In this note we give an example of how a computer algebra system can offer surprises even in the context of a standard calculus topic. When introducing the formula for arc length, some natural examples are the curves C_n which are given parametrically by $x = t^n$, $y = t^{n+1}$, $0 \le t \le 1$, (n is a positive integer). Many students have difficulty computing even the length of C_1 by hand, so this is a natural place to use a computer algebra system. The length of C_5, for example, is

$$\frac{3431\sqrt{61}}{20736} + \frac{15625}{124416}\ln 5 - \frac{15625}{124416}\ln(-6 + \sqrt{61}).$$

As n increases, the results become increasingly unpleasant until, surprisingly, we find that the length of C_{20} is rational and equals $\frac{36495661067145135829027}{25798674916142804999323}$.

It is easy to see why this is so and to show that infinitely many of the lengths $L(C_n)$ are rational numbers. Using the standard formula $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$ for arc length, we have

$$L(C_n) = \int_0^1 t^{n-1}\sqrt{n^2 + (n+1)^2 t^2}\,dt.$$

After we make the successive substitutions $u = \frac{n+1}{n}t$ and $v = \sqrt{u^2+1}$, the integrand becomes $(v^2-1)^{(n-2)/2}v^2$.

We now take n to be even, say $n = 2k$. Then our integrand is just a finite sum of integer multiples of powers of v; that is,

$$(v^2-1)^{(n-2)/2}v^2 = \sum_{j=1}^{k} c_{2j}v^{2j}.$$

for some integers $c_2, c_4, \ldots, c_{2k}$. Integrating and substituting back, we find that

$$L(C_n) = \frac{n^{n+1}}{(n+1)^n}\sum_{j=1}^{k}\frac{c_{2j}}{2j+1}(u^2+1)^{(2j+1)/2}\Big|_0^{n+1},$$

and hence that

$$L(C_n) = \frac{n^{n+1}}{(n+1)^n}\sum_{j=1}^{n/2}\frac{c_{2j}}{2j+1}\left(\left(\frac{\sqrt{(n+1)^2+n^2}}{n}\right)^{2j+1} - 1\right).$$

This will be rational if $n^2 + (n+1)^2$ is a perfect square; that is, if n and $n+1$ are part of a Pythagorean triple. It is well known that there are infinitely many such pairs (see, for example [**1**, p. 164, Exer. 17]). In particular, if $(a, a+1, c)$ is a Pythagorean triple, so is $(3a+2c+1, 3a+2c+2, 4a+3c+2)$. Note that the parity of the first term switches, so that by using this result twice we can go from one even case to another. Consequently, not only does C_{20} have a rational length, so does C_{696}, and also infinitely many other curves C_n.

References

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On the Convergence of Some Modified p-Series

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The series we consider, which we call (S, p)*-series*, are obtained from the p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ by removing all those terms in which the base n in the denominator contains a digit in a specified set S. For example, if $S = \{1, 2\}$, then our series is

$$\frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{9^p} + \frac{1}{30^p} + \frac{1}{33^p} + \frac{1}{34^p} + \cdots + \frac{1}{40^p} + \frac{1}{43^p} + \frac{1}{44^p} + \cdots.$$