Supersolids: Solids Having Finite Volume and Infinite Surfaces

By WILLIAM P. LOVE

A curious and interesting class of geometric solids exists that I have named supersolids. A supersolid is a bounded solid that has finite volume and infinite surface area. A bounded solid is one that may be contained inside a sphere having finite radius.

Supersolids are interesting because they defy our intuitive sense of reality. How can an object have a finite volume and have an infinite surface area? Such paradoxical concepts fascinate students. If it were physically possible to construct a supersolid, one could create a planet using a finite amount of material and have unlimited surface area. This planet might solve the population problem, since everyone could have an infinite amount of land.

The study of supersolids is useful for the first-year calculus teacher. These solids are definitely nonroutine and are rarely seen in any calculus textbook. Because of their curious properties, students are highly motivated to investigate these solids to show that their volume is finite and their area is infinite. The more advanced and creative students may be encouraged to discover new and unusual supersolids of their own.

The traditional calculus curriculum introduces the concept of integration before the concept of infinite series. Students use integration techniques to find the areas and volumes of solids of revolution and other solids. Students are frequently confused when introduced to infinite series because they see no concrete need or application for them. Supersolids furnish an ideal introduction to this topic because students can visualize a situation requiring infinite series. Supersolids are therefore useful for combining two major concepts in calculus: integration and infinite series.

Five examples of supersolids are presented here:

- The supercone
- The supercube
- The superpyramid I
- The superpyramid II
- The supersine solid

The first four are solids generated using elementary geometry and infinite series. Only a few basic properties of infinite series are required, and the concepts are easily understood by most students. The last example is a solid of revolution, which requires a number of integration principles as well as infinite series.

The Supercone

The supercone is formed by taking an infinite sequence of right triangles on the interval [0, 2] and revolving them around the x-axis to form an infinite sequence of right circular cones as shown in figure 1. Let T_1, T_2, $T_3, \ldots, T_n$ represent the infinite sequence of triangles and let C_1, C_2, $C_3, \ldots, C_n$ represent the corresponding infinite sequence of cones formed when the triangles are revolved around the x-axis. The supercone is the union of these cones, which we denote $\sum C_n$.

Define the triangle T_n to have the following:

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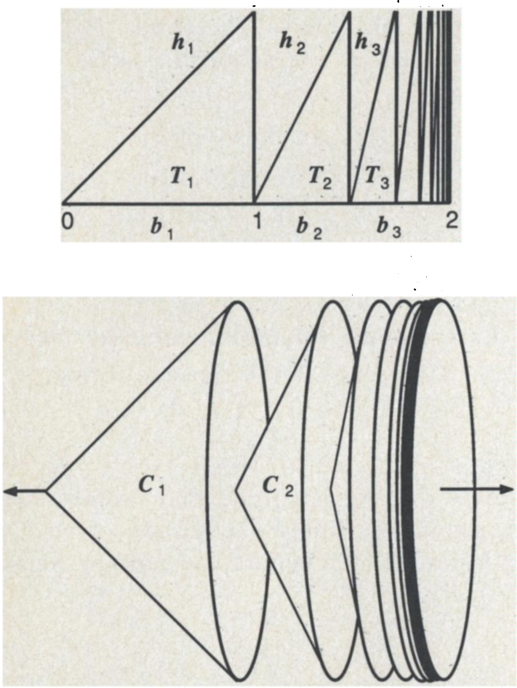


Fig. 1. Supercone: An infinite sequence of right triangles revolved around the x-axis generating right circular cones

Base length, b_n, $= 1/2^{n-1}$
Altitude $= 1$
Hypotenuse, h_n, $= \sqrt{1^2 + b_n^2} = \sqrt{1 + 1/2^{2n-2}}$

Therefore, C_n, the right circular cone formed when T_n is revolved around the x-axis, will have the following volume and lateral surface area:

$$\begin{aligned}\text{Volume} &= 1/3 \cdot (\text{area base})(\text{altitude})\\ &= 1/3 \cdot \pi \cdot \text{radius}^2 \cdot \text{altitude}\\ &= 1/3 \cdot \pi \cdot 1^2 \cdot b_n\\ &= 1/3 \cdot \pi \cdot 1/2^{n-1}\end{aligned}$$

$$\begin{aligned}\text{Surface area} &= \text{area base} + \text{lateral area}\\ &= \pi \cdot \text{radius}^2 + \pi \cdot \text{radius} \cdot \text{slant height}\\ &= \pi \cdot 1^2 + \pi \cdot 1 \cdot h_n\\ &= \pi \cdot (1 + \sqrt{1 + 1/2^{2n-2}})\end{aligned}$$

The following argument proves that the supercone, $\sum C_n$, has finite volume and infinite surface area. Clearly, it is bounded.

$$\begin{aligned}\text{Volume of supercone} &= \sum_{n=1}^{\infty} \text{volume of } C_n\\ &= \sum_{n=1}^{\infty} 1/3 \cdot \pi \cdot 1/2^{n-1}\\ &= \pi/3 \sum_{n=1}^{\infty} 1/2^{n-1}\\ &= \pi/3 \cdot 2\\ &= 2\pi/3\end{aligned}$$

Hence, the supercone has finite volume.

$$\begin{aligned}\text{Area of supercone} &= \sum_{n=1}^{\infty} \text{area of } C_n\\ &= \sum_{n=1}^{\infty} \pi \cdot (1 + \sqrt{1 + 1/2^{2n-2}})\\ &\geq \sum_{n=1}^{\infty} \pi \cdot (1 + \sqrt{1})\\ &= 2\pi \sum_{n=1}^{\infty} 1,\end{aligned}$$

which is infinite. Hence, the supercone has infinite surface area.

The Supercube

The supercube is formed by drilling an infinite sequence of cylindrical holes in a cube with edge equal to one unit. The holes decrease in size but become more numerous as they get smaller as shown in figure 2. The

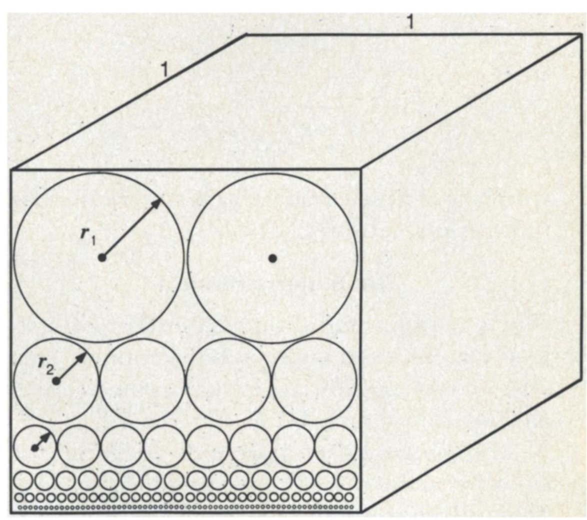


Fig. 2. Supercube: Drill an infinite sequence of holes in a cube having edge equal to one unit.

radii $r_1, r_2, r_3, \ldots, r_n$ of the holes drilled in the cube are defined by $r_n = 1/2^{n+1}$. Each cylindrical hole with radius r_n has a lateral surface area,

$$S_n = 2\pi \cdot r_n \cdot 1 = \pi/2^n.$$

Notice that two holes have radius r_1, four holes have radius r_2, and 2^n holes have radius r_n.

The following argument proves that the supercube has finite volume and infinite surface area. Clearly, it is bounded.

$$\begin{aligned}\text{Volume} &= \text{volume of cube} \\ &\quad - \sum_{n=1}^{\infty} \text{volume of holes} \\ &\le \text{volume of cube} \\ &\le 1\end{aligned}$$

Hence, the supercube has finite volume.

$$\begin{aligned}\text{Surface area} &= \text{area of 4 square faces} \\ &\quad + \text{area of 2 ends with holes} \\ &\quad + \sum_{n=1}^{\infty} \text{lateral area of all holes} \\ &\ge \sum_{n=1}^{\infty} \text{lateral area of all holes} \\ &= \sum_{n=1}^{\infty} (\text{number holes with radius } r_n)(\text{lateral area hole}) \\ &= \sum_{n=1}^{\infty} 2^n \cdot 2\pi r_n \\ &= \pi \sum_{n=1}^{\infty} 2^{n+1} \cdot 1/2^{n+1} \\ &= \pi \sum_{n=1}^{\infty} 1,\end{aligned}$$

which is infinite. Hence, the supercube has infinite surface area.

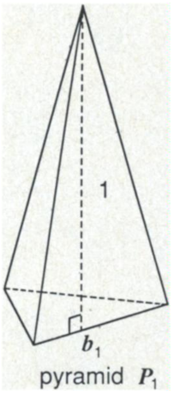
The Superpyramid I

The superpyramid I is the union of infinitely many triangular-based pyramids constructed on an equilateral triangular region as shown in figure 3. Let $P_1, P_2, P_3, \ldots, P_n$ be the sequence of pyramids forming the superpyramid I. Define pyramid P_n to have an equilateral triangular base whose length is $b_n = 1/2^n$ and three triangular faces (excluding base) with the altitude of each face equal to one unit. Therefore, P_n has a volume and a lateral surface area as follows:

$$\begin{aligned}\text{Volume} &= 1/3 \cdot \text{area base} \cdot \text{altitude pyramid} \\ &= 1/3 \cdot (\sqrt{3}/4)b_n^2 \cdot \text{altitude pyramid} \\ &= 1/3 \cdot \sqrt{3}/2^{2n+2} \cdot \text{altitude pyramid} \\ &\le 1/3 \cdot \sqrt{3}/2^{2n+2} \cdot 1 \\ &= 1/\sqrt{3} \cdot 1/2^{2n+2}\end{aligned}$$

$$\begin{aligned}\text{Lateral area} &= 3 \cdot (\text{area triangular face}) \\ &= 3 \cdot (1/2 \cdot \text{base} \cdot \text{altitude}) \\ &= 3 \cdot (1/2 \cdot b_n \cdot 1) \\ &= 3 \cdot 1/2^{n+1}\end{aligned}$$

Notice that we have one P_1 pyramid, three P_2 pyramids, nine P_3 pyramids, and (3^{n-1}) P_n pyramids. We denote the superpyramid I by $\sum P_n$.



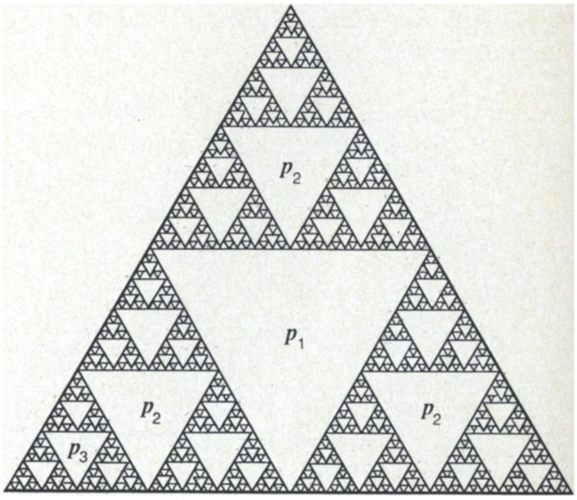


Fig. 3. Superpyramid I: The union of infinitely many triangular based pyramids constructed on an equilateral triangular region

The following argument proves that $\sum P_n$ has finite volume and infinite surface area. Clearly, it is bounded.

$$\text{Volume} = \sum_{n=1}^{\infty} \text{volume of all } P_n$$
$$= \sum_{n=1}^{\infty} (\text{number of pyramids } P_n) \cdot (\text{volume single } P_n)$$
$$= \sum_{n=1}^{\infty} (3^{n-1})(\text{volume } P_n)$$
$$\leq \sum_{n=1}^{\infty} (3^{n-1})(1/\sqrt{3} \cdot 1/2^{2n+2})$$
$$= 1/\sqrt{3} \cdot \sum_{n=1}^{\infty} 3^{n-1}/2^{2n+2}$$
$$= 1/16\sqrt{3} \cdot \sum_{n=1}^{\infty} (3/4)^{n-1}$$
$$= 1/16\sqrt{3} \cdot 4 = 1/4\sqrt{3},$$

which is finite. Hence, the superpyramid I has finite volume.

$$\text{Area} = \sum_{n=1}^{\infty} \text{surface area of all } P_n$$
$$= \sum_{n=1}^{\infty} (\text{number of pyramids } P_n) \cdot (\text{area single } P_n)$$
$$= \sum_{n=1}^{\infty} (3^{n-1})(3/2^{n+1})$$
$$= 1/2 \cdot \sum_{n=1}^{\infty} (3/2)^n,$$

which is infinite. Hence, the superpyramid I has infinite surface area.

The Superpyramid II

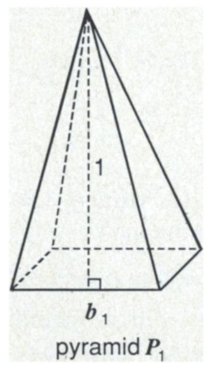
The superpyramid II is the union of infinitely many square-based pyramids constructed on a 1×1 square region as shown in figure 4. Let $P_1, P_2, P_3, \ldots, P_n$ be the sequence of pyramids forming the superpyramid II. Define pyramid P_n to have a square base whose length is $b_n = 1/3^n$ and four isosceles triangular faces (excluding base) with the altitude of each face equal to one unit. Therefore, P_n has the following volume and lateral surface area:

$$\text{Volume} = 1/3 \cdot \text{area base} \cdot \text{altitude pyramid}$$
$$= 1/3 \cdot b_n^2 \cdot \text{altitude pyramid}$$
$$\leq 1/3 \cdot 1/3^{2n} \cdot 1$$
$$= 1/3^{2n+1}$$

$$\text{Lateral area} = 4 \cdot (\text{area triangular face})$$
$$= 4 \cdot (1/2 \cdot \text{base} \cdot \text{altitude})$$
$$= 4 \cdot (1/2 \cdot 1/3^n \cdot 1)$$
$$= 2/3^n$$

Notice that we have five P_1 pyramids, twenty P_2 pyramids, eighty P_3 pyramids, and $(5 \cdot 4^{n-1})$ P_n pyramids. We denote the superpyramid II by $\sum P_n$.

The following argument proves that $\sum P_n$ has finite volume and infinite surface area. Clearly, it is bounded.



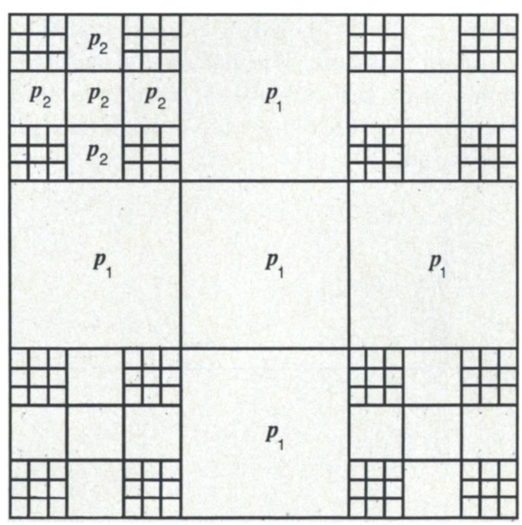


Fig. 4. Superpyramid II. The union of infinitely many square-based pyramids constructed on a 1×1 square region

$$\text{Volume} = \sum_{n=1}^{\infty} \text{volume of all } P_n$$
$$= \sum_{n=1}^{\infty} (\text{number pyramids } P_n) \cdot (\text{volume single } P_n)$$
$$\leq \sum_{n=1}^{\infty} (5 \cdot 4^{n-1})(1/3^{2n+1})$$
$$= \sum_{n=1}^{\infty} (5/27)(4/9)^{n-1}$$
$$= (5/27)(9/5) = 1/3,$$

which is finite. Hence, the superpyramid II has finite volume.

$$\text{Area} = \sum_{n=1}^{\infty} \text{surface area of all } P_n$$
$$= \sum_{n=1}^{\infty} (\text{number of pyramids } P_n) \cdot (\text{area single } P_n)$$
$$= \sum_{n=1}^{\infty} (5 \cdot 4^{n-1})(2/3^n)$$
$$= 10/3 \cdot \sum_{n=1}^{\infty} (4/3)^{n-1},$$

which is infinite. Hence, the superpyramid II has infinite surface area.

The Supersine Solid

The supersine solid is formed by revolving the trigonometric function $f(x) = \sin(1/x)$ around the x-axis over the interval $(0, 2/\pi]$. Figure 5 shows a graph of $f(x) = \sin(1/x)$ over this interval. The following argument proves that the supersine solid has finite volume and infinite surface area. Clearly, it is bounded.

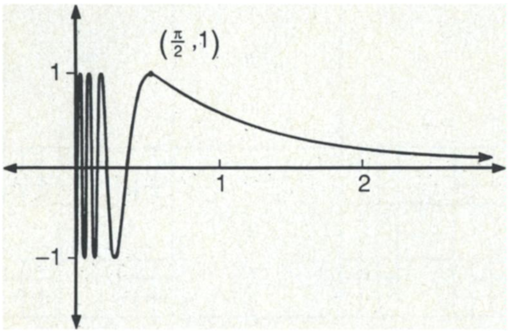


Fig. 5. Supersine solid: Revolve $f(x) = \sin(1/x)$ around the x-axis over interval $(0, 2/\pi]$.

$$\text{Volume} = \text{volume of } f(x) \text{ revolved around } x\text{-axis over } (0, 2/\pi]$$
$$\leq \text{volume of cylinder with radius of 1 over } [0, 2/\pi]$$
$$= 2,$$

which is finite. Hence, the supersine solid has finite volume.

The surface area of the supersine solid can be found by using the integral for surfaces of solids of revolution found in most elementary calculus texts.

$$\text{Area} = \int_a^b 2\pi |f(x)| \sqrt{1 + [f'(x)]^2}\, dx$$

where $f(x) = \sin(1/x)$ over interval $(0, 2/\pi)$.

Area

$$= \int_0^{2/\pi} 2\pi |\sin(1/x)| \sqrt{1 + (1/x^4)\cos^2(1/x)}\, dx$$

By letting $u = 1/x$, $du = -(1/x^2)\, dx$, so that $dx = -du/u^2$; and by replacing the bounds $x = 0$ and $x = 2/\pi$ with $u = \infty$ and $u = \pi/2$ and using the fact that

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx,$$

we get

$$\text{Area} = 2\pi \int_{\pi/2}^{\infty} |\sin u| \sqrt{(1/u^2)^2 + \cos^2 u}\, du$$
$$\geq 2\pi \int_{\pi/2}^{\infty} |\sin u| \sqrt{\cos^2 u}\, du$$
$$= 2\pi \int_{\pi/2}^{\infty} |\sin u| \cdot |\cos u|\, du$$
$$= \pi \int_{\pi/2}^{\infty} |\sin 2u|\, du.$$

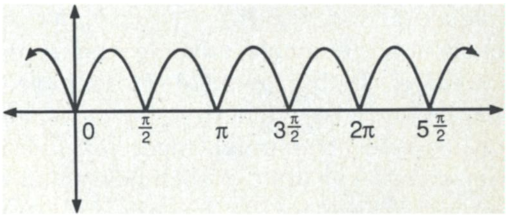


Fig. 6. Graph of $y = |\sin 2u|$

Figure 6 shows the graph $y = |\sin 2u|$. Observe that the pattern over the interval $[\pi/2, \pi]$ is repeated infinitely many times. Thus, by symmetry we can write

$$\text{Area} = \pi \sum_{n=1}^{\infty} \left\{ \int_{\pi/2}^{\pi} |\sin 2u| \, du \right\}.$$

Since $\sin 2u$ is negative over the interval $[\pi/2, \pi]$, we replace $|\sin 2u|$ with $-\sin 2u$ to get

$$\begin{aligned} \text{Area} &= \pi \sum_{n=1}^{\infty} \left\{ \int_{\pi/2}^{\pi} -\sin 2u \, du \right\} \\ &= \pi/2 \sum_{n=1}^{\infty} \{[\cos 2u]_{\pi/2}^{\pi}\} \\ &= \pi/2 \sum_{n=1}^{\infty} \{[\cos 2\pi - \cos \pi]\} \\ &= \pi/2 \sum_{n=1}^{\infty} \{2\}, \end{aligned}$$

which is infinite. Hence, the supersine solid has an infinite surface area.

Conclusion

Students find the study of supersolids interesting and educational. Once they understand the principles, they can create numerous examples of supersolids. Most solids can be transformed into supersolids if you "wrinkle" their surface enough. Fractal curves can be revolved to form fractal solids. Supersolids lend an excellent opportunity to encourage creative and original thinking in mathematics, as well as using some calculus techniques.

It should be noted that all bounded solids must have a finite volume. However, it is fun to prove that their volume is finite. In addition, a supersolid planet would not necessarily solve the population problem, since an infinite area may be covered by a single footprint.

Having investigated solids with a finite volume and infinite surface, students might consider the problem of solids having finite surface area and infinite volume. Such a solid is found in a science fiction novel by Greg Bear (1985). Good luck.

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