Could the ideas of T.S. Kuhn revolutionise mathematics teaching?

Terence Mills
La Trobe University, Vic
<t.mills@latrobe.edu.au>

Aimé Sacrez
La Trobe University, Vic,
<a.sacrez@latrobe.edu.au>

The authors identify paradigm shifts that have occurred in mathematics, and their relevance to teaching mathematics. They argue that T. S. Kuhn's ideas could be used to raise awareness that mathematics adapts to different cultural and historical contexts.

School students might be forgiven for thinking that once mathematical ideas are fixed, they are fixed forever, that Mathematics is right or wrong, and the answers are always in the back of the book. Yet, mathematicians know that this is not the whole story. Thomas Kuhn (1962/2012) introduced the term "paradigm shift" to the scientific literature to describe how knowledge in science develops. The aims of this paper are to identify paradigm shifts, or revolutions, that have occurred in mathematics, and to discuss their relevance to teaching mathematics in our schools. We argue that Kuhn's ideas could be used to promote awareness that mathematics adapts to different cultural and historical contexts.

In the next section, we give a brief outline of the key ideas in Kuhn (1962/2012). Then, we consider its relevance to the development of mathematics. Have there been revolutions in mathematics? We go on to explore links between Kuhn's ideas and Piaget's theory of constructivism. We propose five topics from the *Australian Curriculum: Mathematics* that could be regarded as revolutionary in the sense of Kuhn: the Hindu-Arabic numeral system, algebra, irrational numbers, coordinate geometry, and infinity. We then suggest a sequence of lessons aimed at presenting one of these topics to middle years and senior secondary classes. Conclusions are presented in the final section.

Kuhn's work

Kuhn (1962/2012) describes the process by which knowledge in science develops. Figure 1 is a representation of his model. The process begins with an assortment of ideas circulating in a scientific community that are relevant to the issue at hand. These ideas are debated, and various points of view are formed. After some time, there is general agreement on a paradigm, or a set of "universally recognised scientific achievements that for a time provide model problems and solutions to a community of practitioners" (Kuhn, 1962/2012, p. xlii). Having reached this consensus, scientists then pursue "normal science" (Kuhn, 1962/2012, p. 24). At this stage, scientists do not need to think about the overarching paradigm; instead they focus on solving specific problems. In a sense, they have a reduced vision leading to specialised books, journals, and conferences.

Occasionally, an anomaly is noticed; something does not quite fit in with the paradigm. In an extreme case, the paradigm needs serious revision. We have a crisis. The anomaly is then thrown in with what remains of the assorted ideas. More debate ensues, we wait for a new paradigm to emerge, and we witness a paradigm shift. Kuhn regards this as a revolution (Lauden 1977; Bird, 2000, 2013; Hoyningen-Huene 1993).

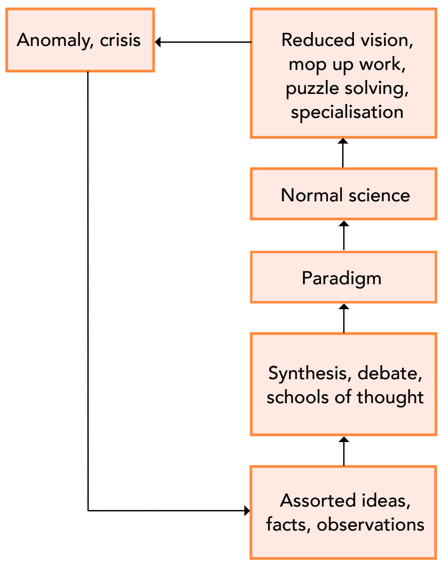

Figure 1. Kuhn's model for the development of scientific knowledge.

Kuhn and mathematics

Kuhn was trained as a physicist; he had no formal training in history or philosophy. When Kuhn talks about science, he is referring principally to physics, chemistry, and, to a lesser extent biology: he feels that he does not have sufficient expertise in biology (Kuhn, 1962/2012, p. xliii). On the other hand, he acknowledges that many ideas in Fleck (1979), which is a translation of a work published in German in 1935, anticipate his own (Kuhn, 1962/2012, p. xli), and Fleck was a biochemist. It seems that Kuhn regards mathematics as separate from science (Kuhn, 1962/2012, p. 15; Hoyningen-Huene, 1993, pp. 4–7).

In this paper, we use the term "mathematics" to mean "pure mathematics". Our view is that, although mathematics is useful in scientific disciplines, it is not part of science.

Nevertheless, Kuhn's book provides a springboard for thinking about the development of mathematics. The connection between Kuhn and mathematics was noticed by Wilder (1968/2013, p. 91). Claire Parkinson, in discussing Kuhn and mathematics, avoided the term "revolution" (Parkinson, 1987, p. 129). Grattan-Guinness (1992) preferred the term "convolution". However, "revolution" is such a good word that we will use it too.

Are there revolutions in mathematics? This question is debated in Gillies (1992). Michael Crowe (1975), which is reprinted in Gillies (1992), proposes ten laws for describing the development of knowledge in mathematics; Wilder (1968/2003, pp. 199-201) offered ten different laws. Crowe made his point of view clear in his tenth law: "Revolutions never occur in mathematics" (Crowe, 1975, p. 165). Of course, this depends on one's definition of "revolution", an issue which is constantly revisited in Gillies (1992). Our view is that there have been discoveries in mathematics that changed the way in which mathematicians viewed the subject and, in this sense, could be regarded as revolutionary, and some of these arise in the *Australian Curriculum: Mathematics.*

Kuhn and constructivism

Kuhn's theory presents knowledge creation as a form of constructivist learning within and between communities of scientists through shifting paradigms (Riegler, 2011). This is evident in Kuhn's statements that "the proponents of competing paradigms practice their trades in different worlds" and "in some areas they see different things, and they see them in different relations one to the other" (Kuhn, 1962/2012, p. 149). While Kuhn's notion of "incommensurability" may not directly apply to mathematical paradigms (1962/2012, p. 147), the historical development of mathematics has involved paradigm transitions and transformations (Mehrtens, 1976, p. 302; Parkinson, 1987, p. 129). Our view is that creating, and then normalising, an alternate paradigm requires a revolutionary shift for mathematicians. This has implications for constructivist mathematics education in fostering knowledge-construction over knowledge-transmission (Cobb,1988; Jaworski, 1994).

According to Kuhn, during periods of "normal science", constructed theory articulates and extends the accepted knowledge of the paradigm (Kuhn, 1962/2012, p. 24). Then, during scientific revolutions, prior theory is reconstructed and "the scientist's perception of his environment must be re-educated" (Kuhn, 1962/2012, p. 112). It is during these periods of paradigm change, that science reveals itself to be a human construction as the scientist's "world of his research will seem, here and there, incommensurable with the one he had inhabited before" (Kuhn, 1962/2012, p. 112). We will show that Kuhn's descriptions of science as a historically constructivist endeavour have a parallel in constructivist theory of learning. Yet, translating Kuhn's historiography of science into a constructivist learning theory requires answering the following question. Is the process of historical development of knowledge similar to how children learn to construct knowledge over time?

This question underpinned the formation of Jean Piaget's genetic epistemology (Kitchener, 1986). Piaget explains "The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological processes" (Piaget, 1970). This allowed the philosophy of constructivism to contribute to the psychology of learning, which has significantly influenced the structure of education.

Kuhn and Piaget's constructivist theories

Interpreting Kuhn's historiographic study of paradigms as a learning theory would seem to lend itself to Vygotsky's (1978) cultural-historical psychology. Yet, we regard Piaget's theory of cognitive development through adaptation to be most applicable to Kuhn's explanation of scientific development through revolutions, which is a view shared by Tsou (2006). Piaget's constructivism not only demonstrates that a parallel can be drawn between the historical development of knowledge for the child and for mankind, it also reveals that the process proceeds through revolutionary stages of "reorganization of knowledge" involving a "total reinterpretation of the conceptual foundations" (Piaget & Garcia, 1989, p. 109).

The theories of Piaget and Kuhn support a renewed focus on the experience of learning for students, which has implications for constructivist teaching of mathematics. Piaget notes that through social interaction and direct experiences, children come to see situations from other perspectives, "decentration" (Piaget, 1976, p. 21). This process initially involves cognitive conflict, "disequilibrium", as new information cannot be assimilated into the learner's current mental representation, "schema" (Brainerd, 1978, p. 282). This requires a restructure of the schema, "accommodation", to regain cognitive equilibrium. Thus, Piaget's theory of cognitive development through adaptation for the individual and Kuhn's revolutions for scientific communities describe parallel growth cycles. See Ernest (1991, pp. 103–104) for more discussion of Piaget and Kuhn.

Table 1 compares terms used by Kuhn to describe the development of scientific knowledge with terms used in Piaget's constructivist learning theory. Although Kuhn's terms are not well-defined in Structure (Lauden, 1977, pp. 73–76), it is a striking parallel.

Table 1. Comparing concepts used by Kuhn with those used by Piaget.

Kuhn	Piaget
Paradigm: Accepted examples or models of practice, including law, theory, application and instruments. Also called a disciplinary matrix.	**Schema:** Structure of a person's existing knowledge, such as concepts, assimilated experiences and actions.
Normal science: Research that fits into existing paradigm.	**Assimilation:** New information fits into existing schema.
Crisis: Technical and theoretical breakdown of the paradigm due to an anomaly.	**Disequilibrium:** Need to restructure schema based on incompatible new information.
Revolution: Turning points in science where old paradigm is replaced by new paradigm.	**Accommodation:** Significant change to schema, leading to cognitive growth.

Piaget and Kuhn provide insights for constructivist learning in showing that learning is a process of construction and reconstruction. Perhaps, at times presenting mathematics as developing through revolutions could increase authenticity and engagement in learning for students. In this way, students not only learn about mathematical revolutions, but also learn through experiencing mathematical revolutions.

Kuhn's theory of paradigms and revolutions could contribute to mathematics education by fostering an appreciation of the importance of periods of crisis for learning. Both Piaget and Kuhn demonstrate that, periods of crisis require the "reconstruction of prior theory" (Kuhn, 1962/2012, p. 7). Piaget sums up thus: "it's just when you have the courage to see that something doesn't work that you find new things" (Bringuier, 1980, p. 115).

Kuhn and mathematics education

Some topics in school mathematics might have been revolutionary when they were introduced to mathematics. We now identify five such topics chosen from different levels throughout the school curriculum.

Hindu-Arabic numerals. Students learn about the Hindu-Arabic numerals in primary school. Fibonacci promoted the Hindu-Arabic numerals, and the associated arithmetic, to Europeans in his famous work Liber Abaci in 1202 (Sigler, 2002). Up until then, the system of Roman numerals prevailed in Europe. Although it is obvious to us that the Hindu-Arabic system of numerals is far superior to the system of Roman numerals, selling this new idea was not all plain sailing. For example, in 1299, bankers in Florence were forbidden from using the Hindu-Arabic system of numerals. Around the same time, prices of books in Padua were to be recorded in words not symbols (Smith & Karpinski, 1911, p. 133; Seife, 2000, p. 80). Neugebauer wrote, "The invention of this place value notation is undoubtedly one of the most fertile inventions of humanity" (1969, p. 5). The introduction of the Hindu-Arabic numerals to medieval Europe has all the hallmarks of a revolution in the sense of Kuhn.

Algebra. Many students find the first encounter with algebra an important point in their mathematical development. In a very interesting paper, Jurdak (1997) discusses the development of algebra in terms of Kuhn's Structure.

Irrational numbers. Students encounter irrational numbers in lower secondary school. Their discovery in ancient Greece caused a stir and could well qualify as a revolution in the sense of Kuhn, and the revolution continues today (Havil, 2012).

Analytical geometry. Students encounter analytical geometry around Year 10. René Descartes (1596-1650) was a major player in the development of analytical geometry (or Cartesian geometry or coordinate geometry) in mathematics. The idea of applying techniques in algebra to problems in geometry was revolutionary because it opened up a radically new approach to thinking about geometrical problems. For discussion on whether analytical geometry can be regarded as revolutionary see Mancosu (1992) and Hacking (2014, pp. 3–6).

Infinity. Australian students encounter infinity in calculus in senior secondary years. Over the centuries, scholars have wondered about the infinite. It was not until the late nineteenth century that Georg Cantor (1845-1918) developed the modern mathematical concept of infinity. Cantor's views were revolutionary. Eli Maor writes, "Needless to say, to express such bizarre views in the nineteenth century was no less than an act of rebellion" (Maor, 1991, p. 56). David Hilbert wrote, "Aus dem Paradise, das Cantor uns gestaffen, soll uns niemand vertrieben können" (Hilbert, 1926): No one will drive us from the paradise that Cantor has created. Indeed, Cantor "opened a new era of mathematics" (Aczel, 2000, p. 111) and a new paradigm about infinity emerged.

School texts seem to rush through these topics, paying little or no attention to their profound, revolutionary nature. Perhaps not enough attention is given to historical-cultural aspects of mathematics in Australian universities.

We will now return to considering the Hindu-Arabic system of numerals to provide a practical illustration of introducing a revolution into mathematics teaching.

Outline of a sequence ofactivities

In this section, we outline a sequence of lessons focussed on numerals that could work for students from Year 4 to 5. This is only a sketch because we have not trialled it in a classroom.

Activity 1: Drawn to numerals

Learning focus: This is the first of three activities on the historical development of numeral systems. Students work in groups of 2 to 3. By using three different numeral systems, students will come to appreciate that numeral systems are creative solutions to problems of everyday mathematics for different cultures and societies.

Objectives:

1. Students will identify some differences and similarities between three historical numeral systems such as Egyptian, Roman and Hindu-Arabic.
2. Students will recognise that numeral systems can have different structures and notations, by applying the numeral system to represent different numbers.
3. Students will distinguish between number and numerals.

Process: Have separate stations for each numeral system with visual examples of the system, some background history of the culture. Students rotate in groups, copy the notation system and answer mathematical questions such as the following: How do you write 307? How do you add and subtract?

Activity 2: Create a numeral system

Learning focus: This is the second of three activities on the historical development of numeral systems. Students will compare the three systems of numerals using criteria for evaluation. Examples include efficiency of notation and simplicity of arithmetic, as well as aesthetics. Challenging students to invent an alternate system will allow for a deeper understanding of numeral systems.

Objectives:

1. Students will evaluate the strength and weakness of each numeral system by example and reasoning.
2. Students will create their own numeral system.
3. Students will begin drafting a presentation poster of their numeral system.

Process: Begin with a class discussion on the merits of each system. Develop criteria for evaluation. Brainstorm creative ideas of other notational forms such as colours, symbols, words, sounds and movements. Students can also change the base, such as: Base 12, Base 2, Base 60. Encourage students' numeral system to be creative and functional.

Activity 3: How does this new system work?

Learning focus: This is the last of three activities on the historical development of numeral systems. Students demonstrate their creative and critical reasoning skills in mathematics by exhibiting their own work, discussing the work of others and reflecting on the topic of numeral systems as shifting paradigms.

Objectives:

1. Students finish their posters and display them. Peer feedback is given based on three criteria: aesthetics, simplicity of notation, and functionality of arithmetic (simpler terms can be used for students such as: looks cool/ beautiful, easy to write, and easy to calculate).
2. Groups of students can explain their system to the class.
3. Students reflect on the reason why the Roman numeral system was replaced.

Process: Students vote for their favourite numeral systems by placing three coloured stickers on posters of their peers. Blue: Aesthetics. Red: Simplicity. Yellow: Functionality. Students are assessed by the teacher based on the mathematical merit of their system, group work, presentation and responses to reflection questions such as: "What determines a successful numeral system?" and "Why did the Hindu-Arabic numeral system replace the Roman numeral system?"

These activity plans demonstrate that a seemingly conventional topic such as numerals can reveal the revolutionary aspects of mathematics. Specifically, students consider a numeral system as a constructed paradigm situated within "the historical integrity of that science in its own time" (Kuhn, 1962/2012, p. 3). Engaging in the "normal science" of each numeral system unearths the crisis of their systems of arithmetic. Yet, they could be dealt with through concepts such as zero; Cajori (1928) discusses how zero has been treated in different cultures. Finally, students experience revolutionary mathematics through the creative task of inventing and critiquing their own numeral systems. Challenging students to construct new paradigms extends their mathematical understanding and reasoning which are key ideas from the *Australian Curriculum: Mathematics* (ACARA, 2010). When textbook exemplars of "normal mathematics" are not sufficient for solving the crisis of unresolved and potentially unresolvable mathematical problems, revolutionary mathematics is needed.

Conclusions

Kuhn (1962/2012) presents a model for describing the development of scientific knowledge. Although his model does not apply directly to the development of mathematical knowledge, it suggests how we can think about the development of mathematics and thereby enhance the teaching of mathematics.

During their years at school, students encounter concepts that were dramatic, perhaps revolutionary, when introduced in mathematics. We have identified several such concepts from primary and secondary years of school. Teachers could give students the opportunity to become aware of the cultural, historical, and revolutionary nature of these ideas. In the words of the *Australian Curriculum: Mathematics*, this will assist students, "to develop an understanding of the relationship between the 'why' and the 'how' of mathematics" (ACARA, 2010).

Perhaps the ideas of Thomas Kuhn can revolutionise mathematics teaching.

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