Question 1 (3 marks)

a. 1 mark

This is just an exercise in completing the square - with nice simple numbers too!

$$p(z) = \left[z^2 + 6iz\right] - 25 = \left[(z+3i)^2 - (3i)^2\right] - 25$$

$$\boxed{p(z) = (z+3i)^2 - 16\text{ , i.e., } a = 3,\ b = -16}$$

b. 2 marks

"Hence" method:

$$0 = (z+3i)^2 - 16 = (z+3i-4)(z+3i+4)$$

$$\Longrightarrow \boxed{z = 4 - 3i \text{ or } z = -4 - 3i}$$

"Otherwise" -- use quadratic equation:

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1)(-25)}}{2 \times 1} = \frac{-6i \pm \sqrt{100 - 36}}{2} = -3i \pm 4$$

Question 2 (3 marks)

Rearrange DE to get $x = \frac{-1}{\sqrt{4-y^2}}\frac{dy}{dx}$

Integrate from $x = 2$ to some x: $\int_2^x x dx = \int_2^x \frac{-1}{\sqrt{4-y^2}}\frac{dy}{dx}dx$

$$\Longrightarrow \left[\frac{1}{2}x^2\right]_2^x = \frac{1}{2}x^2 - 2 = -\int_0^y \frac{1}{\sqrt{4-y^2}}dy = -\arcsin\left(\frac{y}{2}\right) + 0$$

$$\Longrightarrow \boxed{y = 2\sin\left(2 - \frac{1}{2}x^2\right)}$$

Check:

$$y(2) = 2\sin(2-2) = 0$$

$$\frac{dy}{dx} = 2\left(-\tfrac{1}{2}\right)(2x)\cos\left(2 - \tfrac{1}{2}x^2\right) = -2x\sqrt{1 - \sin^2\left(2 - \tfrac{1}{2}x^2\right)}$$

$$= -2x\sqrt{1 - 2^{-2}y^2} = -x\sqrt{4 - y^2}$$ ☺

Question 3 (4 marks)

a. 2 marks

T = random variable for the time to dispense one cup of coffee

$T \sim \mathrm{N}(\mu = 10,\ \sigma = 1.5)$

Let X = the random variable for the time to dispense 4 cups of coffee

Note: $\mu_X = E(X) = 4E(T)$, $\mathrm{Var}(X) = 4\mathrm{Var}(T) \implies \sigma_X = 2\sigma_T$

$X \sim T_1 + T_2 + T_3 + T_4 = \mathrm{N}(\mu = 40,\ \sigma = 3)$

$\Pr(X > 34) = \Pr(X > \mu_X - 2\sigma_X) = \Pr(Z > -2)$, where Z is a standard normal deviate

From the 68-95-99.7 rule, it follows that

$\Pr(Z > -2) \approx 0.95 + 0.025 = 0.975$

$\boxed{\Pr X > 34) \approx 0.98}$

b. 2 marks

$n = 25,\ \hat{y} = 9$

$$CI = \hat{y} \pm z_* \sigma_{\hat{Y}} = 9 \pm 1.96\, \frac{1.5}{\sqrt{25}} \approx 9 \pm 2 \times \frac{3}{10} = 9 \pm \frac{3}{5}$$

$\boxed{CI = (8.4,\ 9.6)}$

Question 4 (4 marks)

Easy way is to carefully split up the numerator

$$\boxed{\int \frac{3\left(x^2+4\right)+4x}{x\left(x^2+4\right)}dx = \int \frac{3}{x} + \frac{4}{x^2+4}dx = 3\ln|x| + 2\arctan\left(\frac{x}{2}\right) + c,\quad c \text{ is a constant}}$$

Default approach would be to use partial fractions

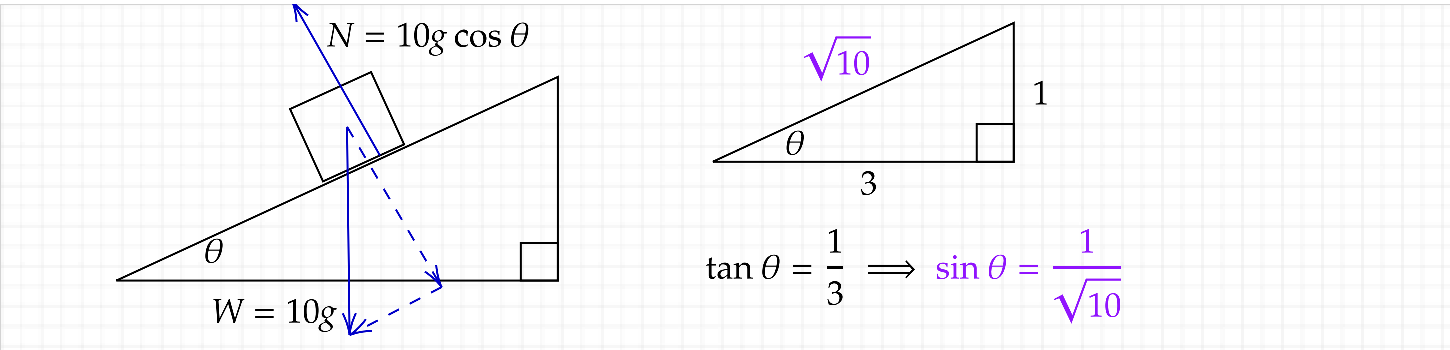
$$\frac{3x^2+4x+12}{x\left(x^2+4\right)} = \frac{A}{x} + \frac{B+Cx}{x^2+4} = \frac{A\left(x^2+4\right)+x(B+Cx)}{x\left(x^2+4\right)} = \frac{(A+C)x^2+Bx+4A}{x\left(x^2+4\right)}$$

Comparing coefficients of x^n in the numerator:

$A = 3,\ B = 4,\ C = 0$

$$\implies \int \frac{3x^2+4x+12}{x\left(x^2+4\right)}dx = \int \frac{3}{x} + \frac{4}{x^2+4}dx = 3\ln|x| + 2\arctan\left(\frac{x}{2}\right) + c,\quad c \text{ is a constant}$$

Question 5 (3 marks)



a. 2 marks

$F_{net} = 10g\sin\theta = 10g \times \frac{1}{\sqrt{10}} = \sqrt{10}\,g$

$a = \frac{F_{net}}{m} = \frac{g}{\sqrt{10}}$. This is a constant acceleration, so can use SUVAT, $u = 0,\ t = 2,\ v = ?$

$v = u + at = 0 + \frac{g}{\sqrt{10}} \times 2 \implies \boxed{v = \frac{\sqrt{10}g}{5}}$

b. 1 mark

The braking force exactly cancels the net force (component of the weight down the plane) found above, $R = \sqrt{10}g$ up the plane (against the motion).

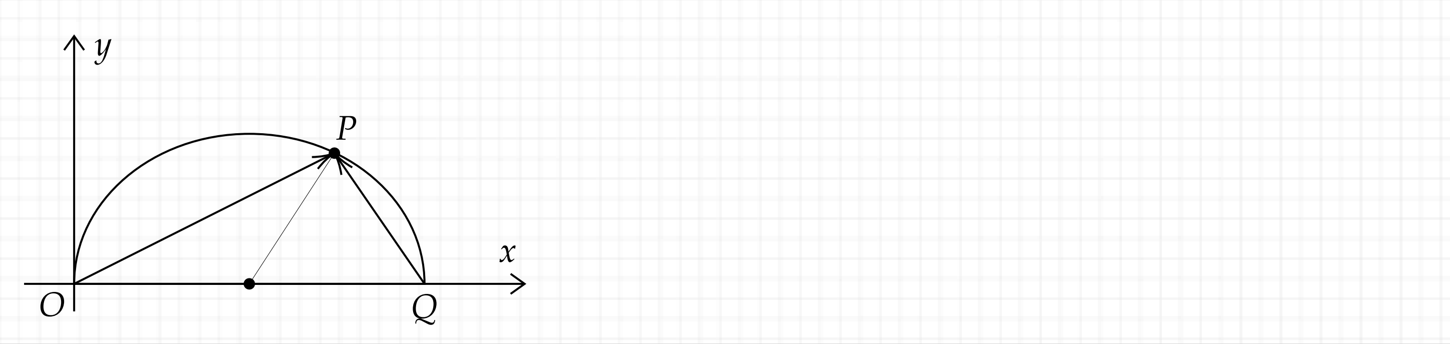
Question 6 (6 marks)

a. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ 2 marks

Let θ be the angle between **a** and **b**

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{2-6+12}{\sqrt{4+9+36}\sqrt{1+4+4}} = \frac{8}{7\times 3} \implies \boxed{\cos\theta = \frac{8}{21}}$$

b. OPQ is a semicircle of radius a with equation $y = \sqrt{a^2-(x-a)^2}$.



i. 1 mark

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$
$$\overrightarrow{QP} = \overrightarrow{QO} + \overrightarrow{OP} = -2a\mathbf{i} + x\mathbf{i} + y\mathbf{j} = (x-2a)\mathbf{i} + y\mathbf{j}$$

ii. 3 marks

$$\overrightarrow{OP}\cdot\overrightarrow{QP} = x(x-2a) + y^2 = x^2 - 2ax + y^2$$

But, x, y are on the circle, so $y^2 = a^2 - (x-a)^2 = 2ax - x^2$

$$\overrightarrow{OP}\cdot\overrightarrow{QP} = x^2 - 2ax + \left(2ax - x^2\right) = 0$$

So, the angle between $\overrightarrow{OP}$ and $\overrightarrow{QP}$ is $90°$ and the two vectors are perpendicular.

(This is just a vector proof of Thales' theorem)

Question 7 (3 marks)

Differentiate both sides of the equation:

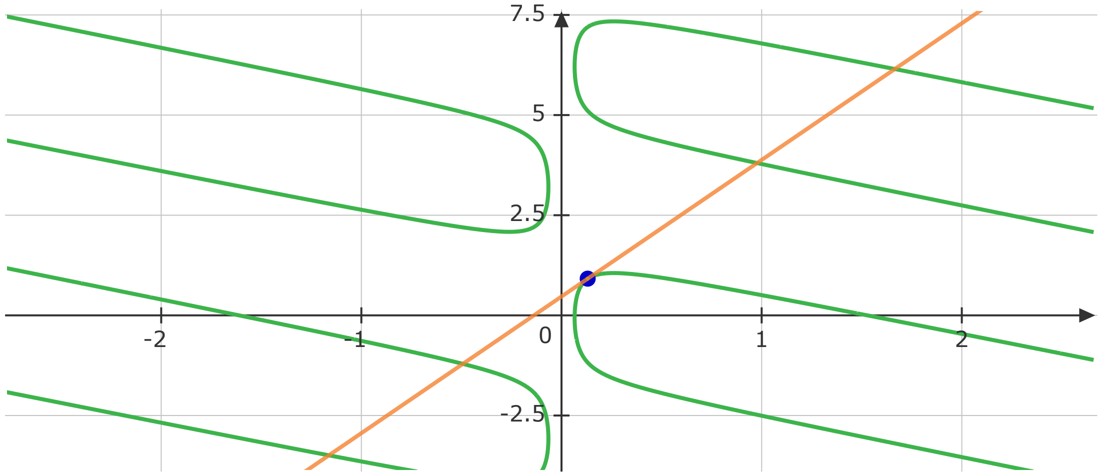
$$0 = \frac{d}{dx}\frac{\pi}{48} = \frac{d}{dx}(x\cos(x+y)) = \cos(x+y) + x\left(1+\frac{dy}{dx}\right)(-\sin(x+y))$$

$$\implies 0 = \cos(x+y) - x\sin(x+y)\left(1+\frac{dy}{dx}\right)$$

$$\implies \frac{dy}{dx} = \frac{\cos(x+y)}{x\sin(x+y)} - 1 = \frac{1}{x}\cot(x+y) - 1$$

At the given point: $\dfrac{dy}{dx} = \dfrac{24}{\pi}\cot\left(\dfrac{\pi}{3}\right) - 1 = \dfrac{24}{\pi}\dfrac{1}{\sqrt{3}} - 1 \implies \boxed{\dfrac{dy}{dx} = \dfrac{8\sqrt{3}-\pi}{\pi}}$

It is actually a pretty nice periodic curve - pictured below, including the tangent at the given point



Question 8 (4 marks)

$a = -4x = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$\implies c - 2x^2 = \dfrac{1}{2}v^2$, c is a constant

When $x = 0,\ v = -2$, so $c = 2$

$\implies v^2 = 4\left(1 - x^2\right)$

$\implies \boxed{v = -2\sqrt{1 - x^2}}$

Note that the velocity is negative during the specified integral.

This is just a mass on a spring being released from a positive displacement until it reaches its largest negative displacement.

Question 9 (4 marks)

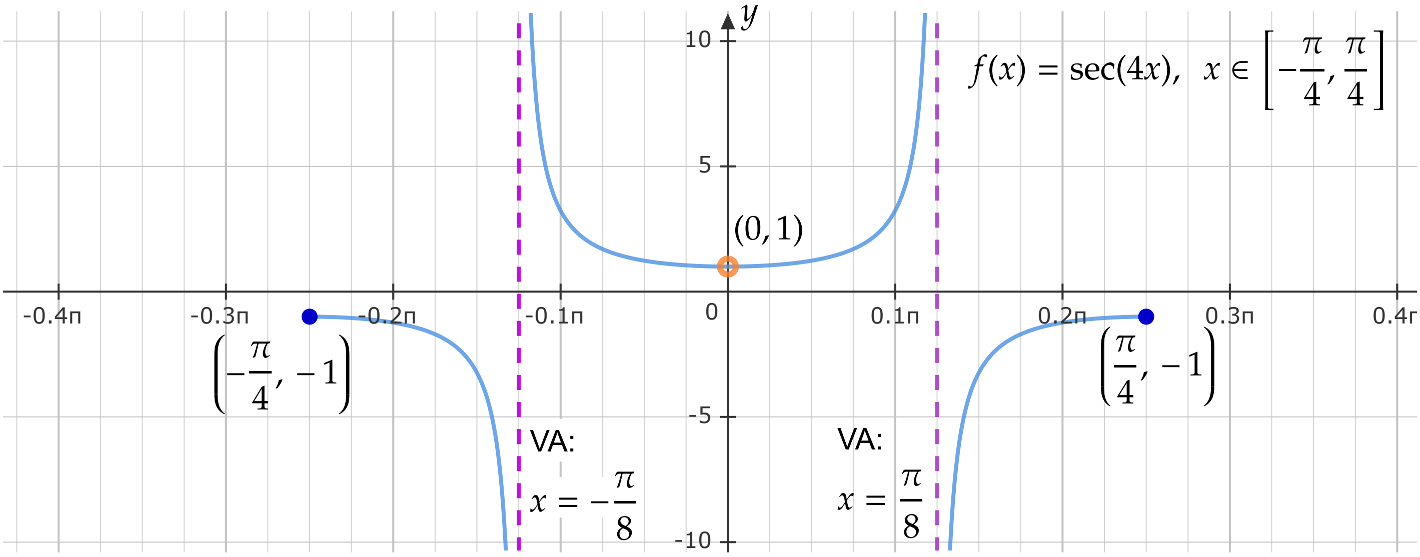
$f(x) - f\left(\dfrac{\pi}{8}\right) = \displaystyle\int_{\frac{\pi}{8}}^{x} f'(x)dx = \int_{\frac{\pi}{8}}^{x} \dfrac{\cos(2x)}{\sin^3(2x)}dx$, let $u = \sin(2x),\ du = 2\cos 2x\, dx$

$\implies f(x) = \dfrac{3}{4} + \dfrac{1}{2}\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\sin(2x)} \dfrac{1}{u^3}du = \dfrac{3}{4} - \dfrac{1}{4}\left[u^{-2}\right]_{\frac{1}{\sqrt{2}}}^{\sin 2x} = \dfrac{3}{4} - \dfrac{1}{4}\left(\dfrac{1}{\sin^2 2x} - \dfrac{1}{1/2}\right)$

$\implies \boxed{f(x) = \dfrac{1}{4}\left(5 - \csc^2 2x\right)}$ or equivalent expression

Question 10 (6 marks)

a. 3 marks



b. 3 marks

$$V = \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \pi\, y^2 dx = \int_{-\frac{\pi}{24}}^{\frac{\pi}{48}} \pi \sec(4x)^2\, dx = \left[\frac{\pi}{4}\tan(4x)\right]_{-\frac{\pi}{24}}^{\frac{\pi}{48}} = \frac{\pi}{4}\left(\tan\left(\frac{\pi}{12}\right) - \tan\left(-\frac{\pi}{6}\right)\right)$$

Need $\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ and $\tan\left(\frac{\pi}{12}\right)$, which is a bit more annoying...

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{4}\right)} = \frac{\sqrt{3} - 1}{1 + \sqrt{3} \times 1} = \frac{\left(\sqrt{3} - 1\right)^2}{2} = 2 - \sqrt{3}$$

$$\Longrightarrow \boxed{V = \frac{\pi}{4}\left(\left(2 - \sqrt{3}\right) + \frac{\sqrt{3}}{3}\right) = \frac{\left(3 - \sqrt{3}\right)\pi}{6}}$$

Of course, can also have any a and b you want and $V = \frac{\left(a - \sqrt{b}\right)\pi}{c}$, provided $c = 6\,\frac{a - \sqrt{b}}{3 - \sqrt{3}}$...