Unsanctioned Corrections to VCAA Implementation Material – Proof by Contradiction

Example 3: Continuation from $e^a = 5^b$. (*)

Option 1 continuation:

$$\Rightarrow \underbrace{e \times e \times \ldots e}_{a \text{ times}} = \underbrace{5 \times 5 \times \ldots 5}_{b \text{ times}}.$$

But this is a falsehood since $e < 5$ and 5 is not divisible by e (**).

The assumption that $\log_e(5)$ is rational has led to a falsehood therefore $\log_e(5)$ is irrational.

Option 2 continuation:

Now 5^b is an integer therefore e^a must be an integer and so $e^a - k = 0$ for some $k \in Z^+$.

But e is a transcendental number and so, by definition, there is no integer polynomial with finite terms that has e as a root (***).

Therefore we have a contradiction in the fact that e is transcendental.

Therefore $\log_e(5)$ is irrational.

* After the line $e^a = 5^b$, it is **not** correct to immediately conclude a contradiction by making a casual remark such as:

> "*It's clear at this point that the right hand side must be divisible by five but the left hand side is not.*"
>
> VCAA support material for implementation of the VCE Mathematics Study Design (2023-2027) in schools: Proof by Contradiction transcript.

Such remarks essentially assume the conjecture that we are attempting to prove.

** The falsehood can also be established by extending the *Fundamental Theorem of Arithmetic* (also called the *unique factorisation theorem* or *prime factorisation theorem*) (which states that every integer greater than 1 can be represented uniquely as a product of prime numbers, up to the order of the factors) to real numbers. However, this would require further proof.

*** This requires students to know a basic fact about transcendental numbers, which is arguably outside the scope of the VCE Study Design. In which case, the VCAA Example 3 is poorly chosen. Proving the irrationality of $\log_2(3)$ would be far more appropriate.