Proof by contrapositive

Recall that $P \Rightarrow Q$ means: if P is true, then Q must also be true. (at least, that's one interpretation.)

So, if Q is false, then P must also be false.

In symbols:

$$(\sim Q) \Rightarrow (\sim P).$$

This expression is called the *contrapositive* of $P \Rightarrow Q$.

They are logically equivalent.

So rather than proving $P \Rightarrow Q$, we can prove $\sim Q \Rightarrow \sim P$ instead.

Let's try our proof again.

$\sim Q \Rightarrow \sim P$

$P \Rightarrow Q$

For any $n \in \mathbb{Z}$, if n^2 is even then n is even.

$\forall n \in \mathbb{Z}\ (n^2 \text{ even}) \Rightarrow (n \text{ even}).$

Proof:

We use proof by contrapositive, and we'll prove

$\forall n \in \mathbb{Z}\ (n \text{ odd}) \Rightarrow (n^2 \text{ odd}).$

Let $n \in \mathbb{Z}$ be arbitrary.

Assume that n is odd.

Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$ by defn. of odd.

So, $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2k_2 + 1$ where $k_2 = 2k^2 + 2k \in \mathbb{Z}$.

$\therefore \exists k_2 \in \mathbb{Z}$ s.t. $n^2 = 2k_2 + 1$.

$\therefore n^2$ is odd. by defn. of odd.

We've proved that if n is odd then n^2 is odd.

By contrapositive, if n^2 is even then n is even.

Since n was arbitrary, this is true for all $n \in \mathbb{Z}$.

$\square$