

2023 Mathematical Methods Exam 2 MCQ 9

Saturday, 4 November 2023 8:50 PM

Question 9

The function f is given by

$$f(x) = \begin{cases} \tan\left(\frac{x}{2}\right) & 4 \leq x < 2\pi \\ \sin(ax) & 2\pi \leq x \leq 8 \end{cases}$$

The value of a for which f is continuous and smooth at $x = 2\pi$ is

A. -2

B. $-\frac{\pi}{2}$

C. $-\frac{1}{2}$

D. $\frac{1}{2}$

E. 2

Continuity:

$$\tan(\pi) = \sin(2\pi a) = 0 \Rightarrow 2\pi a = m\pi, m \in \mathbb{Z} \\ \Rightarrow a = \frac{m}{2} \quad \text{--- (1)}$$

Smooth:

$$\frac{1}{2\cos^2(\pi)} = a\cos(2\pi a) \Rightarrow 1 = 2a\cos(2\pi a) \quad \text{--- (2)}$$

$$\text{Sub (1): } 1 = m\cos(m\pi),$$

$$\cos(m\pi) = \pm 1,$$

$$\text{Case 1: } \cos(m\pi) = 1 \Rightarrow m = 1 \text{ inconsistent } \therefore \text{reject.}$$

$$\text{Case 2: } \cos(m\pi) = -1 \Rightarrow m = -1 \text{ consistent.}$$

Therefore $a = -\frac{1}{2}$ Option C is the intended answer.

check

$$f(x) = \begin{cases} \tan\left(\frac{x}{2}\right), & 4 \leq x < 2\pi \\ \sin\left(-\frac{x}{2}\right), & 2\pi \leq x \leq 8 \end{cases}$$

check of continuity: $\left. \begin{aligned} \lim_{x \rightarrow 2\pi^+} f(x) &= \lim_{x \rightarrow 2\pi^+} \sin(-\frac{x}{2}) = 0 \\ \lim_{x \rightarrow 2\pi^-} f(x) &= \lim_{x \rightarrow 2\pi^-} \tan(\frac{x}{2}) = 0 \end{aligned} \right\} \Rightarrow \lim_{x \rightarrow 2\pi} f(x) = 0$

$$f(2\pi) = \sin(-\pi) = 0.$$

Therefore $\lim_{x \rightarrow 2\pi} f(x) = f(2\pi)$

Therefore continuous at $x = 2\pi$.

check of 'smoothness':

Require $f'(x)$ to be continuous at $x = \frac{\pi}{2}$. — VCAA definition of 'smooth'

$$f'(x) = \begin{cases} \frac{1}{2\cos^2(\frac{x}{2})}, & 4 < x < 2\pi \\ -\frac{1}{2}\cos(-\frac{x}{2}), & 2\pi \leq x < 8 \end{cases}$$

Either $<$ or $\leq$. The calculations below will decide.

$$\textcircled{1} \quad \lim_{x \rightarrow 2\pi^+} f'(x) = \lim_{x \rightarrow 2\pi^+} f'(x) \left(= \frac{1}{2} \right)$$

and therefore $\lim_{x \rightarrow 2\pi} f'(x)$ exists.

$\textcircled{2}$ $f(x)$ is continuous at $x = 2\pi$.

$\textcircled{1}$ and $\textcircled{2} \Rightarrow f'(x)$ is continuous at $x = 2\pi$ (proof is beyond the scope of Units 3 and 4 mathematics and uses the 'Mean Value Theorem').

And so $f'(x) = \begin{cases} \frac{1}{2\cos^2(\frac{x}{2})}, & 4 < x < 2\pi \\ -\frac{1}{2}\cos(-\frac{x}{2}), & 2\pi \leq x < 8 \end{cases}$

* We note that $f'(x)$ is defined at $x=2\pi$ but this does not imply that $f'(x)$ is continuous at $x=2\pi$.

since $\lim_{h \rightarrow 0} \frac{f(2\pi+h) - f(2\pi)}{h}$ exists:

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(2\pi+h) - f(2\pi)}{h} &= \lim_{h \rightarrow 0^+} \frac{\sin(-\pi - \frac{h}{2}) - \sin(-\pi)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\sin(\frac{h}{2}) - 0}{h} \\ &= \frac{1}{2} \lim_{h \rightarrow 0^+} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} = \frac{1}{2}(1) \\ &= \left(\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f(2\pi+h) - f(2\pi)}{h} &= \lim_{h \rightarrow 0^-} \frac{\tan(\pi + \frac{h}{2}) - \sin(-\pi)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{\tan(\frac{h}{2})}{h} = \lim_{h \rightarrow 0^-} \frac{\sin(\frac{h}{2})}{h \cos(\frac{h}{2})} \\ &= \lim_{h \rightarrow 0^-} \frac{1}{\cos(\frac{h}{2})} \lim_{h \rightarrow 0^-} \frac{\sin(\frac{h}{2})}{h} \\ &= 1 \times \frac{1}{2} = \left(\frac{1}{2}\right) \end{aligned}$$