2022 Specialist Mathematics Exam 2, Section B Question 6 part (f)

Solution.

We define the following random variables:

C: 'mass of an empty can (grams)'.

It is given that $C \sim \text{Normal}\left(\mu_C = 15,\ \sigma_C = 0.25\right)$[1]

T: 'mass of a can filled with soft drink (grams)'.

It is given that $T \sim \text{Normal}\left(\mu_T = 406,\ \sigma_T = 5\right)$[1]

V: '**volume** of soft drink in a can (ml)'.

L: '**mass** of soft drink in a can (grams)'

and $L = 1.04V$ since "1 mL of soft drink has a mass of 1.04 grams."[1]

$\Pr(V < 375) = \Pr(L < 390)$ is required.

The calculation requires knowing the probability distribution function (pdf) of *L*. The pdf can be found in two different ways:

(1) $L = T - C$.

$$\mu_L = E(L) = E(T - C) = E(T) - E(C) = 406 - 15 = 391.$$

To calculate $\text{Var}(L)$, the assumption that *T* and *C* are independent is required[2]. Then:

$$\text{Var}(L) = \text{Var}(T - C) = \text{Var}(T) + \text{Var}(C) = 5^2 + (0.25)^2 = 25.0625$$

$$\Rightarrow \sigma_L = \sqrt{\text{Var}(L)} = 5.00625.$$

Since *T* and *C* are normal it follows that $L \sim \text{Normal}\left(\mu_L = 391,\ \sigma_L = 5.00625\right)$.

Therefore $\Pr(L < 390) = 0.420838$.

The assumption that *T* and *C* are independent is made in the Examination Report. However, this is **not** the assumption stated in the VCAA Defence. Of greater concern is the statement

"This question was interpreted as intended by the Study Specialist Vetter (SSV) and Examination Sitter Reviewer (ESV)."

in the VCAA Defence. This can only mean that the comments made in the Examination Report are wrong.

(2) $T = C + L$.

$$E(T) = E(C + M) = E(C) + E(L) \Rightarrow 406 = 15 + E(L) \Rightarrow E(L) = \mu_L = 391.$$

To calculate $\text{Var}(L)$, the assumption that *C* and *L* are independent must be made.[3] Then:

$$\text{Var}(T) = \text{Var}(C + L) = \text{Var}(C) + \text{Var}(L) \Rightarrow \text{Var}(M) = \text{Var}(T) - \text{Var}(C)$$

$$\Rightarrow \sigma_L^2 = \sigma_T^2 - \sigma_C^2 = 5^2 - (0.25)^2 = 24.9375$$

$$\Rightarrow \sigma_L = \sqrt{\sigma_L^2} = 4.99375.$$

Since T and C are normal it follows that $L \sim \text{Normal}\left(\mu_L = 391,\ \sigma_L = 4.99375\right)$.

Therefore $\Pr(L < 390) = 0.420642$.

We note that this answer is **different** to the answer obtained in (1). This is because a different assumption is used. It is noted that both answers are the same when rounded to three decimal places.

The assumption that C and L are independent is stated in the VCAA Defence. However, this is **not** the assumption stated in the Examination Report. There is a direct contradiction between the VCAA's two official statements about this question.

The following concerns naturally arise:

i) The question cannot be answered unless an assumption is made. Regardless of how reasonable the assumption is, it's not the job of the student to have to make an assumption.

ii) Two reasonable assumptions can be made and each one leads to a different calculation. Was full credit given to correct calculations for each assumption in the marking of the exam?

iii) The VCAA Defence is in direct contradiction to the Exam Report.

Notes:

1. 2022 VCAA Specialist Mathematics Exam 2 (page 23).

2. Justification of this assumption is **not** obvious from the physical situation. It is made purely out of mathematical expediency: the required calculations cannot be performed without it. This assumption is made in the Examination Report without comment.

3. Justification of this assumption is superficially reasonable from the physical situation, as noted in the VCAA Defence:

"Students are meant to realise that a machine dispensing varying amounts of fluid will have no connection to, and be independent from, another machine which makes cans of varying mass at the separate sites."

However, if (for example) "a machine fills a can until a specified total mass is achieved then the amount of fluid dispensed will depend upon the masses of the empty cans, and thus upon the machine that "completed" the cans." (https://mathematicalcrap.com/2023/12/15/vcaas-defence-of-the-indefensible-2022-exam-questions/)