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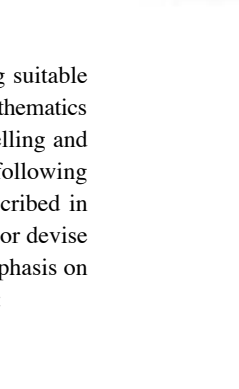
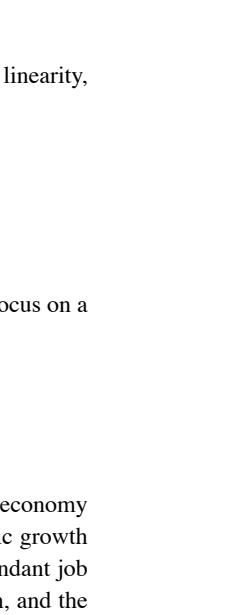
SELECTIONS FROM VCE SEASON OF EXCELLENCE





VCAA Bulletin Supplement 2

VCE Mathematics application tasks

Introduction

The following suggested themes and related advice are provided to assist teachers in devising suitable application tasks for Further Mathematics Unit 3, Mathematical Methods Units 3 and Specialist Mathematics Unit 4 in 2005. Application tasks are particularly well suited to the use of investigative, modelling and problem solving approaches that involve the use of mathematics in real life contexts. The following suggested themes and contexts conform to the design parameters for an application task described in the *VCE Mathematics Assessment Guide*. Teachers may use the starting points outlined below, or devise their own application tasks. All outcomes are to be covered by the application task, with an emphasis on Outcomes 2 and 3. The three components of the application task for each course are as follows:

Further Mathematics – a data analysis application task

1. Displaying and organising univariate and bivariate data.
2. Consideration of general features of the data.
3. Undertaking analysis of the data such as regression analysis, the use of transformations to linearity, de-seasonalisation or analysis of time series.

Mathematical Methods – a function and calculus application task, and Specialist Mathematics – a problem-solving or modelling application task

1. Introduction of a context through specific cases or examples.
2. Consideration of general features of this context.
3. Variation, or further specification, of assumptions or conditions involved in the context to focus on a particular feature related to the context.

Themes and contexts

Further Mathematics – Theme: Growth and sustainability

Robust economic growth is generally considered to be a positive indicator, conversely, if the economy is not growing or is contracting, this is usually viewed with some concern. A lack of economic growth is often depicted in the popular media by images of industries and businesses closing and attendant job losses. Economic growth is typically associated with increased individual and common wealth, and the increased production of consumer goods and availability of services. In many cases this is related to continued, and sometimes increased, use of resources such as oil, minerals, forestry products, and other natural resources.

Victoria The Place To Be

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A topical issue is the increasing concern by some about the ***sustainability*** of our natural resources and environment. Related questions are whether economic growth is leading to the degradation and loss of natural resources for future generations, for example, will future generations have the same access to fresh water and clean oceans, wilderness areas, fuel supplies, biodiversity, and the host of other environmental conditions that have been enjoyed by past generations and are currently enjoyed. What is the relationship between a continued emphasis on economic growth, resource use and possible global climate change connected to the greenhouse effect?

The use of fossil resources such as coal, oil or gas is a major component in the production of consumables. As countries have developed, economic growth has generally led to a greater output of carbon dioxide emissions from the burning of the fossil fuels. Carbon dioxide (CO_2) forms the largest proportion of emissions into the atmosphere contributing to the greenhouse effect. The greenhouse effect is contributing to global warming and potential climate change with a wide range of possible implications such as melting of the ice caps and rising of sea levels, and increased occurrence of droughts, floods and famines. While there is ongoing debate about the relationship between economic growth and sustainability, it is certainly the case that this relationship is the subject of research and policy considerations.

The **Kyoto Protocol** attempts to bring countries of the world together in agreement to limit carbon dioxide emissions and hence reduce the impact of global warming. The consideration of what is a suitable balance between the benefits of economic growth, environmental costs and sustainability is a significant issue for both developed and developing nations. In this investigation, students consider the relationships between CO_2 emissions data and economic prosperity as measured by Gross National Income. Trends in the data can also be considered.

Three related data sets have been provided for this application task sourced from the World Bank website <http://devdata.worldbank.org/hnpstats/query/default.html>. They are available, in Microsoft Excel format at <www.vcaa.vic.edu.au/vce/studies/mathematics/further/furthermathindex.html>.

Data set 1:
GNI (Gross National Income per capita, in $US) and Carbon Dioxide emissions (tons per capita) for five regions of developing countries (East Asia and the Pacific, Europe and Central Asia, Latin America and the Caribbean, Middle East and North Africa, South Asia, Sub-Saharan Africa), the World, Australia, and The United States of America from 1970 to 2002.

Data set 2:
Carbon Dioxide emissions for *High Income*, *Medium Income*, and *Low Income* earning countries across the world, from 1970 to 2002.

Data set 3:
Carbon Dioxide emissions for 160 countries for 1990 and 2000.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- the standard statistical terms and techniques used to display, summarise and describe univariate data for numerical data
- the concept of sample and population and the use of random numbers as a means of selecting a simple random sample of data from a population
- the standard terms and techniques used to display and describe associations in bivariate data for numerical data
- the technique of regression as a means of modelling the relationship between two numerical variables with a straight line
- the role of residual analysis and the coefficient of determination in making decisions about the appropriateness of a particular regression model
- the concept of data linearisation through transformation
- the terms used to describe standard patterns in time series in qualitative terms, the role of smoothing in helping to identify these patterns, and some simple techniques for quantifying these patterns
- the assumptions and/or limitations that underlie the applications of statistical techniques.

All aspects of Outcome 2 and Outcome 3 are relevant.

Starting point 1: Investigation of relationships between Carbon Dioxide emissions and Gross National Income per capita (*Data set 1*)

In developing countries economic growth is substantially generated by the burning of fossil fuels and high levels of resource use. This results in increased levels of CO_2 emissions into the atmosphere. The link between CO_2 levels and the Greenhouse effect is well recognised.

Investigation of the relationship between Carbon Dioxide (CO_2) emissions and Gross National Income per capita (GNI) for the world and for various regions of developing countries. What is the difference between looking at the whole world and various regions separately? Do the regions of developing countries show the same data patterns as The United States or Australia?

Component 1: Selection of relevant data, construction of scatterplots showing CO_2 and GNI for the world and for the selected regions.

Component 2: Summary and description of the distributions of CO_2 and GNI (univariate analysis looking at shape, centre and spread of the distributions).

Component 3: Use of suitable transformations to linearise the relationships if required. Calculation of correlation coefficients, coefficient of determination, and regression equations for linear relationships. Interpretion and explanation of findings.

Key assessment features

The following key knowledge (with corresponding key skills) for Outcome 1, in addition to that listed under the theme, is particularly appropriate for this starting point:

- the standard statistical terms and techniques used to display, summarise and describe univariate data for both categorical and numerical data
- the concept of sample and population and the use of random numbers as a means of selecting a simple random sample of data from a population
- the standard terms and techniques used to display and describe associations in bivariate data for both categorical and numerical data
- the technique of regression as a means of modelling the relationship between two numerical variables with a straight line
- the role of residual analysis and the coefficient of determination in making decisions about the appropriateness of a particular regression model
- the concept of data linearisation through transformation
- the assumptions and/or limitations that underlie the applications of statistical techniques.

Important aspects of mathematics to be considered in assessment of student work are:

- use a range of standard statistical techniques and terms to display, summarise, describe and interpret patterns in data, and outline the assumptions and/or limitations relating to the application of these skills
- use the technique of linear regression to model a relationship between two numerical variables
- interpret the parameters in a regression equation in relation to the situation being modelled
- use one of the listed data transformations, where appropriate, to linearise a set of bivariate data as a means of improving the fit of a regression model.

The data sets could be used to develop some other possible starting points:

Starting point 2: Investigation of growth in CO_2 emissions (*Data set 1*)

The Kyoto protocol took effect in March 1994 and now has now been ratified by 189 countries. The protocol has an 'ultimate objective' of stabilising 'greenhouse gas concentrations in the atmosphere at a level that would prevent dangerous anthropogenic (human-induced) interference with the climate system'. The protocol sets targets for the gradual reduction of levels of CO_2 emissions into the atmosphere by signatory countries.

Investigation of trends in how CO_2 emissions have changed over time. Comparison and interpretation of findings.

Component 1: Selection of data from different regions and construction of time-series plots.

Component 2: Description of the data in the plots, summary and description of the distribution of the series.

Component 3: Investigation of a model to best describe the sample and the use of a trend line for predictive purposes and analysis on the fit of the model.

Starting point 3: Investigation of CO_2 emissions for different income groups *(Data set 2)*

The Kyoto protocol places major responsibility for responding to the challenge of climate change, with related costs, on relatively rich and developed countries. It recognises that poorer nations have a right to economic development and that the share of global emissions of greenhouse gases originating in developing countries will grow as these countries expand their industries to improve social and economic conditions for their citizens.

Investigation and comparison of trends in CO_2 emissions for different income groups of countries. Interpretation and explanation of findings.

Component 1: Selection of data from the different income groups and construction of time-series plots.

Component 2: Description of the data in the plots, summary and description of the distribution of the series.

Component 3: Investigation of a model to best describe the sample and the use of a trend line for predictive purposes and analysis on the fit of the model.

Starting point 4: Comparison of CO_2 emissions in 1990 with CO_2 emissions in 2000 *(Data set 3)*

Not all countries have ratified the Kyoto protocol. Some have made commitments to reduce CO_2 emissions, while others have not.

Investigation of the differences between the CO_2 emissions for one hundred and sixty countries in the years 1990 and 2000. identification and description of a model for this bivariate data set.

Component 1: Selection of a sample of data and construction of boxplots to compare the two years.

Component 2: Description of the data in each of the boxplots, and a description of the differences in the data sets.

Component 3: Investigation of the bivariate relationship; finding a model that best describes the sample and the use of a trend line for predictive purposes and analysis on the fit of the model.

Teachers are encouraged to find other data sets and develop their own investigation based around the theme. There are many other indications that perhaps question whether our current models of economic growth will lead to sustainable environmental outcomes. Various economic data sets and issues such as greenhouse warming in Australia, environmental degradation (e.g. salination of fresh waterways or loss of arable land), number of endangered species and trends in rainfall data could be used.

Some possible sources of relevant data are:

1. http://earthtrends.wri.org/
   This is the World Resources Institute's portal to a vast array of environmental information. There are links to various world organisation's websites.
2. www.deh.gov.au/erin/edd/
   The Australian Government's Department of Environment and Heritage environmental data directory. It contains a wealth of environmental data on all types of environmental measures.
3. www.bom.gov.au/climate/averages/
   The Australian Government's Bureau of Meteorology climate averages page. Links to many sources of data.
4. www.ciesin.columbia.edu/indicators/ESI/rank.html
   The Environmental Sustainability Index (ESI) is the result of collaboration among the World Economic Forum's Global Leaders for Tomorrow Environment Task Force, The Yale Center for Environmental Law and Policy, and the Columbia University Center for International Earth Science Information Network (CIESIN). The ESI provides a basis for addressing a number of pressing policy questions, such as: 'Does good environmental performance come at a price in terms of economic success?'
5. http://unfccc.int/resource/convkp.html
   The website of the Convention and Kyoto Protocol provides information on countries that have ratified the treaty and background information on the issues of economic growth and climate change.

Mathematical Methods – Theme: Modelling with combined functions

Starting from basic functions of a single real variable, such as power functions, circular functions and exponential and logarithmic functions, simple transformations of these functions and their graphs can be considered to develop models for a range of practical situations, involving variation, growth and decay, the motion of objects, measures and scales of intensity, such as strength of earthquakes and periodic behaviour such as tidal patterns.

For modelling of more complex behaviour, functions formed from combinations of these basic functions and their transformations are often used. If f and g are two real functions, then a new function h can be defined in terms of the functions f and g in various ways. The domain d_h and rule $h(x)$ of the combined function h will be specified in terms of the domains d_f, d_g and rules $f(x)$, $g(x)$ of the component functions f and g respectively. Consideration also needs to be given to the continuity and differentiability of the combined function in terms of the continuity and differentiability of its component functions.

A common way of combining two functions, f and g, is to form a *linear combination* of them, that is:

$$h(x) = a f(x) + b g(x) \text{ where } a \text{ and } b \text{ are real constants and } d_h = d_f \cap d_g$$

this includes as special cases the sum ($a = 1$, $b = 1$) and difference ($a = 1$, $b = -1$) of two functions. Another common way of combining two functions f and g is to form their *product function*, that is:

$$h(x) = f(x)\, g(x) \quad \text{where } d_h = d_f \cap d_g.$$

Two functions can also be combined by using the output from one of the functions as input for the other function. This process is called function of a function, chain of functions or composition of functions. Given any two functions, f and g, there are two ways in which this composition can take place, $h(x) = f(g(x))$ or $h(x) = g(f(x))$. The resulting combined functions are usually different functions, depending on which function is used first in application to the independent variable. Consideration also needs to be given as to whether the set of outputs from the first function, that is, its range, will always be a subset of the domain of the second function, thus:

$$h(x) = f(g(x)) \text{ where } r_g \subseteq d_f \quad \text{or} \quad h(x) = g(f(x)) \text{ where } r_f \subseteq d_g$$

In situations where this is not the case, some form of restriction may need to be placed on the domain of the function that is first applied to the independent variable.

Sometimes a combined function h is formed by specifying that it is defined in terms of a collection of component functions, where exactly one of each of component functions applies on given subset of the domain of h. In this case, the function h is called a hybrid function. For example, given two functions f and g which are defined on domain R, a hybrid function h could also be defined on R by:

$$h(x) = \begin{cases} f(x) \text{ for } x < 0 \\ g(x) \text{ for } x \geq 0 \end{cases}$$

An important consideration for the analysis of hybrid functions is the behaviour of the function at the endpoints of the intervals used to define h in terms of its component functions – is h *continuous* or *differentiable* at these interval endpoints?

Where combined functions are used as models in given situations, further consideration may need to be given to the domain and range of both component and combined functions. Practical constraints may require that restrictions are placed on these beyond those required for the combined function to be well defined mathematically. Rate functions for combined functions can also be derived using rules for differentiation of linear combination, product and chain functions.

Some practical situations where combined functions may be employed for modelling purposes are: the amount of a drug in the bloodstream following a therapeutic dose, which involves consideration of a function for absorption into the blood stream along with a function for the removal of the drug from the bloodstream, damped oscillatory motion, logistic population growth and fitting a smooth function to a set of experimentally obtained data points using the method of cubic splines.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- the key features and properties of a given graph of a function or relation or families of types of functions or relations;
- the concepts of domain, range and asymptotic behaviour of functions
- features which enable the recognition of general forms of possible models for data presented in graphical or tabular form;
- analytical or numerical approaches to solving equations and the nature of the corresponding solutions (real, exact or approximate) and the effect of domain restrictions
- features which link the graph of a function to the graph of the corresponding gradient function or its numerical values, the tangent and normal to a curve at a given point, how the sign and magnitude of the derivative of a function can be used to describe key features of the function and its derivative function.

All aspects of Outcome 2 and Outcome 3 are relevant.

Starting point 1: Modelling with sums of functions

Many situations can be modelled using functions which are sums or differences of other functions. Pollution from two or more chimneys (such as the exhaust stacks in a tunnel) can be modelled by combining the pollution generation functions for each separate chimney. Company sales can be modelled by one function giving the general trends and another reflecting seasonal trends. Stock prices can also be modelled by one function giving general trends, another reflecting seasonal variation and yet another reflecting the economic cycle.

Sums of functions can also be used to approximate other functions. Functions which are continuous and have continuous derivatives at a point can be approximated by *Taylor polynomials*, which are simply sums of power functions. Functions which are periodic can be approximated by sums of sine and cosine functions using what are called *Fourier series*.

Component 1: Consideration of a simple sum of functions to model a situation, for example

$$f(x)=\sin(kx)+x,\quad g(x)=1-e^{-kx}-\frac{x}{4} \text{ or } P(x)=\frac{1}{x^2}+\frac{k}{(x-3)^2}$$

and its graph over a suitable domain. Determination of key features of the graph including intercepts, stationary points and equations of asymptotes, as applicable. Consideration of the behaviour of the sum function in terms of the behaviour of its component functions.

Or, *alternatively*: determination of a suitable quadratic function to approximate, for example, $\cos(x)$ at $x = 0$ or $\log_e(1 + x)$ at $x = 0$, where the quadratic function must have the same value as the function and its derivative and the derivative of its derivative at the point. Determination of a measure of closeness of fit over an appropriate domain.

Or, *alternatively*: consideration of sums of functions of the form $\sin(nx) + \cos(nx)$, for a particular $n \in Z$. Use of addition of ordinates for determination of the rule for the resulting function, and consideration of periodicity. Extension to sums of the form $a \sin(nx) + b \cos(nx)$, for some $n \in Z$ and $a, b \in R$, using addition of ordinates for conjecture of the rule for the resulting function.

Component 2: Introduction of a parameter into the model and consideration of the effect on features of the graph, such as intercepts, the location of stationary points, equations of asymptotes. For example,

$$f(x)=\sin(x)+x, \text{ or } g(x)=1-e^{-x}-\frac{x}{4}, \text{ or } P(x)=\frac{1}{x^2}+\frac{1}{(x-3)^2}$$

Or, *alternatively*, extension of the approximation to a third degree polynomial, so that the value of the third derivative of the function at a point matches the third derivative of the polynomial. Consideration of closeness of values of polynomial to original function over appropriate domain.

Or alternatively, consideration of sums of functions of the form $\sin(nx) + \sin(mx)$, or $\cos(nx) + \cos(mx)$, or $\sin(nx) + \cos(mx)$, where $m, n \in Z$. Consideration of key features of graphs, including periodicity and symmetry. For example, for the function f where

$$f(x)=24\sin\left(\frac{2\pi x}{5}\right)-5\sin(2\pi x)$$

determination of the period and demonstration that $f(5 - x) = -f(x)$ and $f(2.5 - x) = f(x)$. Use of these properties in description of the graph of $y = f(x)$.

Component 3: Selection of suitable parameters and development of a model for a given scenario. Consideration of appropriate domain and investigation of the behaviour of the model function in terms of systematic variation of the parameters.

Or, *alternatively*: extension of the approximation process to higher degree. Consideration of closeness of fit over appropriate domain or application of the process for another function, such as $\log_e(1 + x)$ or $\frac{1}{1-x}$ about $x = 0$, considering closeness of fit over appropriate domain for different functions.

Or, alternatively: consideration of partial sums of

$$F(x) = \frac{1}{2} + \frac{2}{\pi}\sin(x) + \frac{2}{3x}\sin(3x) + \frac{2}{5\pi}\sin(5x) + \frac{2}{7\pi}\sin(7x) + \ldots$$

or

$$G(x) = \pi + 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \frac{1}{5}\sin(5x) - \ldots\right)$$

or

$$U(t) = \frac{1}{\pi} + \frac{1}{2}\sin(at)$$

$$-\frac{2}{\pi}\left(\frac{1}{1\times 3}(2at) + \frac{1}{3\times 5}\cos(4at) + \frac{1}{5\times 7}\cos(6at) + \frac{1}{7\times 9}\cos(8at) + \ldots\right)$$

or similar as approximations to the square wave function f, with period 2π, given by

$$f(x) = \begin{cases} 0 \text{ if } -\pi \le x < 0 \\ 1 \text{ if } \quad 0 \le x < \pi \end{cases}$$

or the saw-tooth function g, with period 2π, given by

$$g(x) = x + \pi \quad \text{if} \quad -\pi < x < \pi$$

or half-wave rectifier , with period, $\frac{2\pi}{a}$ given by

$$u(t) = \begin{cases} 0 \text{ if } -\frac{\pi}{a} < t < 0 \\ \sin(at) \text{ if } 0 < t < \frac{\pi}{a} \end{cases}$$

respectively. Consideration of goodness of fit. Consideration of value of approximation at midpoint and endpoints of principal domain.

Or, alternatively: investigation of symmetries in the graphs of functions with rules of the of the form $h(x) = a\sin(mx) + b\sin(nx)$, where a, b, m and n are non-zero real constants.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features;
- the use of calculus to determine the derivative of a given function;
- appropriateness of model;
- addition of ordinates to determine/estimate rule of function;
- determination of closeness of fit;
- consideration of domains.

Starting point 2: Modelling with products of functions

Computer drawing packages usually have a *Bezier tool*, which draws a smooth curve between two points, and where the user has some control over the position of this curve. Such curves are specified by determining the coordinates at any point on the curve. The x- and y-coordinates are each determined by cubic polynomial functions, and each of these polynomial functions is a linear combination of the set of product polynomial functions:

$$\{ t^3, 3t^2(1-t), 3t(1-t)^2, (1-t)^3 \}$$

with coefficients of the linear combination determined by control points specified by the user.

Similarly, in a *Bernoulli experiment* with n trials and probability p of success in any trial, the probability of obtaining exactly r successes in the n independent trials is given by ${}^nC_r\, p^r\, (1-p)^{n-r}$. Each of these probabilities is a product of a p^r term and a $(1-p)^{n-r}$ term.

Due to the action of damping forces of various sorts, physical oscillations usually decay unless there is some driving force applied. This decay can be modelled by a product function with tule of the of the form $h(x) = ae^{-x}\sin(kx)$ for suitable values of a and k. The quantity of a drug absorbed into the body may also be modelled by a product function with rule of the form $ax^n e^{-kx}$ for suitable values of a, n and k. This sort of model reflect both the absorption of the drug into the body as well as the breaking down or removal of the drug from the body.

The preceding are examples of where products of functions are used in modelling. This starting point considers a combined function that is a product of simple functions used to model such situations and investigates variations of the model.

Component 1: Selection of a suitable product function h of functions f and g, such as $x^2(1-x)$, xe^{-x}, $e^{-x}\sin(x)$ as a possible model for some situation. Analysis of the graph of the function and identification the key features. Consideration of the graph of h and its key features with respect to the graphs of f and g and their key features.

Component 2: Investigation of families of functions with rules similar to the above with respect to a defining parameter, for example $x^r(1-x)^{n-r}$ for $r, n \in Z$, with $0 \le r \le n$, $x^n e^{-x}$ for $n \in Z$, or xe^{-cx} for $c \in Z$, or $e^{-kx}\sin(x)$ or $e^{-x}\sin(kx)$ for suitable $k \in R$. Identification of key features of these families of product functions and their graphs, including consideration of the graphs of the family of functions and their key features with respect to the graphs of the component functions and their key features.

Component 3: Further variation of parameters, and investigation of features of resulting families of curves for product functions of a given type in terms of their defining parameters and component functions (this could, for example, be extended to investigation of a product function defined in terms of three component functions).

Or, alternatively: investigation of functions used to compute binomial probabilities. Selection of a situation to model, such as the number of sixes in 100 throws of a die, or the number of times black turns up in 100 spins of a roulette wheel, and investigation of the distribution of probabilities for a fixed number of successes, for less than or equal to a fixed number of successes, or for greater than a fixed number of successes.

Or, alternatively: investigation of cubic *Bezier* curves. Given four control points

(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), the coordinates of points on the curve are given by

$$x(t) = x_1t^3 + 3x_2t^2(1-t) + 3x_3t(1-t)^2 + x_4(1-t)^3$$

$$y(t) = y_1t^3 + 3y_2t^2(1-t) + 3y_3t(1-t)^2 + y_4(1-t)^3$$

for $0 \le t \le 1$. Consideration of endpoints (for $t = 0$ and $t = 1$). Consideration of types of curves which can be produced.

Or, alternatively: consideration of a model for decaying oscillator such as a grandfather clock which has not been wound up. Investigation of key features and relevance to the model, such as how long before the oscillations are so small they will not be discernible to the human eye.

Or, alternatively: consideration of a model for drug absorption. Investigation of key features and relevance of the model, such as how much drug can be taken before a critical (fatal) level may be reached, how much time elapses before the drug will be less than a given value.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features;
- the use of calculus to determine the derivative of a given function;
- interpretation of key features of graph in relation to practical situation;
- consideration of domains.

Starting point 3: Modelling using functions of functions

Functions such as e^{-x^2}, $\cos\left(\frac{1}{x}\right)$, $\sin(x^2)$ and $\frac{1}{1+e^{-x}}$ arise in various modelling contexts, and are referred to as functions of functions, chain functions or composite functions. The function with rule $h(x) = e^{-x^2}$, or a suitable transformation of this function, is used to model *errors in measurements* and, suitably scaled, becomes the *standard normal* probability density function for a continuous random variable. This function is a function of a function where the rule of first function applied to the independent variable is $g(x) = -x^2$ and the rule of the second function is $f(x) = e^x$. The *logistic* function

$$P(x) = \frac{1}{1+e^{-x}}$$

is used in modelling population growth where there is a limit to the available resources. Here the first and second functions have the rules $g(x) = 1 + e^{-x}$ and $f(x) = \frac{1}{x}$ respectively. It should be noted that while the domain of the combined function of a function is a subset of the domain of the first function, its range will be a subset of the range of the second function. In modelling contexts, it is also likely that additional restrictions on domain and range will be imposed with respect to conditions associated with the situation under investigation.

Component 1: Selection of a function $h = f(g(x))$ such as one of those listed above, or similar, and consideration of its graph and identification of its key features, in particular the influence of the first function on the resulting graph. Consideration of the derivative and its relation to the graph of the function. For functions f and g consideration of the line segments formed from joining in order the sequence of points $(x, 0)$, $(x, g(x))$

$(g(x), g(x))$, $(g(x), f(g(x)))$), $(x, f(g(x))$ and the locus of points formed in this way.

Component 2: Variation of the function investigated above, by the introduction of parameters corresponding to a transformation of one of the functions. Consideration of key features of resulting family of curves.

Or, alternatively: for,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{(2\sigma^2)}}$$

investigation of the effect of fixing one parameter and systematically varying the other.

Or, alternatively: for a population which can be modelled by,

$$P(x) = \frac{L}{1 + Ae^{-kt}} \text{ where } A = \frac{L - P_0}{P_0}$$

and P_0 is the initial population, investigation of the effect of varying P_0 while fixing L and k . Consideration of the resultant family of curves, including consideration of derivative of P with respect to t.

Or, alternatively: selection of the functions with rules $f(x) = 1 - x$ and $g(x) = \frac{1}{x}$

and consideration of the new functions formed by $f(g(x))$ and $g(f(x))$ Composition of these two new functions with themselves and also with f and g in various combinations. Investigation of the number of *distinct* functions, including f and g themselves, that are formed in this way, and the domain over which *all* of these combinations are defined.

Component 3: Investigation of systematic variation of other parameters in the model and consideration of the effect on key features of the graph of the combined function.

Or, alternatively: application of numerical integration to estimate probabilities related to a normal distribution, for example, the heights of males or females in a given population. Estimation of scores corresponding to a specified probability.

Or, alternatively: application of the logistic model to a particular population. Consideration of key features of the graph for the population, and the rate of change of the population over an appropriate domain.

Or, alternatively: for the functions with rules $f(x) = -x$ and $g(x) = \frac{x+1}{-x+1}$

consideration of the new functions formed as $f(g(x))$ and $g(f(x))$. Composition of these two new functions with themselves and also with f and g in various combinations. Investigation of the number of *distinct* functions, including f and g themselves, that are formed in this way, and the domain over which *all* of these combinations are defined. Extension of the investigation to subfamilies of the set functions with rules of the form $f(x) = \frac{ax+b}{cx+d}$

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions that clearly indicate key features
- the use of calculus to determine the antiderivative of a given function.
- consideration of domains
- numerical integration
- numerical solution of equations.

Starting point 4: Modelling using hybrid functions

Hybrid functions are frequently used to model data when the use of single function, or even a combination of functions such as those considered in the preceding starting points is not suitable. Some examples could include modelling a path along a river, modelling a part of a rollercoaster ride or simply determining the graph of a function that passes through a set of data points. In other instances, such as fixed fees for service charged per time interval, or approximations for numerical integration of functions, step functions provide a natural model. How the component functions of a hybrid function are to be fitted together is important. Depending on the situation, there may be no constraints on how the parts fit together, and in some cases they may not join at all. In other cases, the component functions may need to fit together so as to be continuous, or may need to fit together smoothly in which case the values of both the functions and the derivatives should match at the joining points. They may need to fit together with the same concavity (that is, their graphs 'curve in the same direction'), in which case the values of the functions, their derivatives and the *derivatives of their derivatives* should match at the joining points.

Component 1: Selection and consideration of a simple hybrid function to model one of the above or similar scenarios. For example, an ***S*** bend in a bike path, by fitting two parabolas to each of three selected points; a rollercoaster curve, by fitting a parabola with a line segment joined at each end for a section of a rollercoaster ride. Consideration of key features of the hybrid model and relevant constraints.

Or, alternatively: determination of the family of parabolas that pass through two distinct points, say $(1, a)$ and $(2, b)$ and consideration of key features of the corresponding family of curves.

Component 2: Investigation of selected model, in particular with respect to determination of a set of component functions that have smooth joins, so that the derivative is defined and continuous on the domain of the combined (hybrid) function.

Or, alternatively: inclusion of a third point say (3, c), and for each of the family passing through the first two points, construction of a parabola through the second pair of points so that the derivative at the join is continuous. (Hint: writing the second parabola in the form $f(x) = p + q(x - 2) + r(x - 2)^2$ will facilitate calculations). Consideration of the key features of these joined parabolas to form a combined function whose graph is a smooth curve that passes through the specified points. Consideration of the graphs of the derivative functions and of the derivative function of the derivative functions for the combined function and the corresponding joined parabolas.

Component 3: Further investigation of the model, with respect to selection of a set of component functions for which the corresponding curves are smoothly joined, so that the derivative is defined and continuous on the domain, and so is the derivative function of the derivative function (that is the curve has the same concavity where the graphs of the component functions are joined).

Or, alternatively: investigation of fitting a cubic polynomial function through each pair of consecutive points so that the resultant model is continuous, differentiable and the derivative of the derivative is continuous everywhere (these are called cubic splines). Consideration of constraints.

Key assessment features

Important aspects of mathematics to be considered in assessment of student work for this starting point are:

- sketch graphs of functions and families of functions that clearly indicate key features
- solution of equations
- suitable choice of models
- consideration of domains

Specialist Mathematics – Theme: Difference and differential equations

Difference and *differential* equations can be used to model a range of applications which involve rates of change with respect to time. These models can vary between contexts that have theoretical and practical applications. Consideration can also be given to solutions of difference and differential equations using numerical methods. Related practical application contexts may involve the modelling of chemical reactions, the dispersal of a pollutant through a medium, or the growth and decline of a population.

Key knowledge for Outcome 1 relevant to this theme (with corresponding key skills) would include knowledge of:

- specified functions and relations, the form of their sketch graphs and their key features, including asymptotic behaviour
- techniques for finding derivatives and anti-derivatives of functions, the relationship between the graph of a function and the graph of its anti-derivative functions and definite integrals
- standard modelling contexts for setting up differential equations and associated solution techniques, including numerical methods
- analytical, graphical and numerical techniques for setting up solving equations involving functions and relations.

All aspects of Outcome 2 and Outcome 3 are relevant.

Students may need to draw two- and three-dimensional graphs, where curves are specified by cartesian equations or parametric equations. The use of suitable technology such as graphics calculators, spreadsheets or computer algebra systems will be particularly appropriate for carrying out these processes.

Starting point 1: The rate reaction

The rate at which a chemical reaction occurs is generally governed by the concentrations of the reactants available to the reaction as well as the temperature of the process. In the case of a simple reaction, there might be a single substance A that reacts to form substance B:

$$A \rightarrow B$$

The rate at which this reaction occurs will be a function of time, and may be governed by an equation of the form:

$$\frac{d\left[A(t)\right]}{dt} = -k_1\left[A(t)\right]$$

where $[A(t)]$ represents the concentration of substance A, at time t, and k_1 is the rate constant for the reaction. Square brackets are used in this context to denote the concentration of a chemical reactant. For this simple reaction it can also be noted that:

$$\frac{d\left[A(t)\right]}{dt} = -\frac{d\left[B(t)\right]}{dt}$$

An example of such a process is the radioactive decay of uranium.

Component 1: Consideration of the simple reaction above and determination of a function for $[A(t)]$ using the rate equation, given an initial concentration $[A(0)]$ of substance *A*. Graphical analysis of $[A(t)]$ as a function of *t* for various values of k_1 (for example starting with $[A(0)] = 1$, and $k_1 = 0.001$ s^{-1}, varying the value for k_1 to consider 0.01, 0.002, 0.0005 and 0.0001 s^{-1}).

Component 2: Consideration of a more complex situation where first reaction is immediately followed by a second reaction in which the substance B formed reacts to form substance C:

$$B \rightarrow C$$

where substance B is an intermediate substance whose concentration increases as a result of the decay of substance A, but decreases as a result of its own decay:

$$\frac{d\left[B(t)\right]}{dt} = k_1\left[A(t)\right] - k_2\left[B(t)\right]$$

where k_2 is the rate constant for the second reaction. Determination of a function for $[B(t)]$ using the rate equation, and recognition that $[A(t)] + [B(t)] + [C(t)] = [A(0)]$.

Consideration of the graphs of $[A(t)]$, $[B(t)]$ and $[C(t)]$ plotted on the same set of axes, and their key features, as a function of time (for example, $[A(0)] = 1$, $k_1 = 0.001$ s^{-1} and $k_2 = 0.0002$ s^{-1}). Investigation of the behaviour of the system for different values of k_1 and k_2.

Component 3: Investigation of the set of values for k_1 and k_2 so that the concentration of the intermediate substance never exceeds half the initial concentration of substance *A*, that is

$$\left[B(t)\right] \leq \frac{\left[A(0)\right]}{2}$$

Exploration of the behaviour of the system with respect to other constraints.

Or, alternatively: Investigation of the system involving the reaction:

$$A \rightarrow B \rightarrow C$$

where $\frac{d\left[A(t)\right]}{dt} = -k_1\left[A(t)\right]^2$ and $\frac{d\left[B(t)\right]}{dt} = k_1\left[A(t)\right]^2 - k_2\left[B(t)\right]$

Starting point 2: Smokey's bar

Active and passive smoking in restaurants, bars and other recreational venues has been a topical issue from both individual health and work and safety considerations. Suppose that a bar with a volume of 600 m^3 has 60 smokers in it, each smoking an average two cigarettes per hour. When smoked each cigarette produces a range of chemicals including 1.4 mg of formaldehyde. Formaldehyde converts to carbon dioxide with a reaction rate of 0.40/hr. Assume that fresh air enters the bar at a rate of 1200 m^3/hr and stale air containing some of the formaldehyde and carbon dioxide leaves at the same volumetric flow rate. Fans in the ceiling ensure that the concentrations of the formaldehyde and carbon dioxide are uniform throughout the room.

A differential equation for the amount of formaldehyde in the bar at any given time can be written as:

$$\begin{pmatrix} rate\ of\ accumulation \\ of\ formaldehyde \end{pmatrix} = \begin{pmatrix} Rate\ at\ which \\ formaldehyde \\ enters\ the\ room \end{pmatrix} - \begin{pmatrix} Rate\ at\ which \\ formaldehyde \\ leaves\ the\ room \end{pmatrix} + \begin{pmatrix} Rate\ of \\ formaldehyde \\ generation \end{pmatrix} - \begin{pmatrix} Rate\ of \\ formaldehyde \\ consumption \end{pmatrix}$$

and modelled mathematically by:

$$V_r \frac{dC(t)}{dt} = Q(t)C_o - Q(t)C(t) + G - kC(t)V_r$$

Here V_r is the volume of the room, $C(t)$ is the concentration of formaldehyde, C_o is the concentration of formaldehyde in the air entering the room, *t* is the time elapsed from a fixed starting time, $Q(t)$ is the volumetric flow rate of the air entering the room, *G* is the rate at which formaldehyde is formed by the smoking of the cigarettes, and *k* is the first-order reaction rate.

For Smokey's bar it is known that $k = 0.40$/hr and $C_o = 0$.

Component 1: Consideration of the Smokey's bar scenario and related issues based on solution of the given differential equation. For example, determination of the steady-state concentration of formaldehyde in the room after several hours (the threshold formaldehyde concentration for eye irritation is about 60 mg/m^3). Investigation of the behaviour of the system by systematic variation of parameters and conditions such as the volume of the room, the fresh air volumetric flow rate and the number of smokers. Consideration of the value of *Q* for which the threshold eye irritation concentration is never reached.

Component 2: Consideration of the situation where the generation of formaldehyde varies with time (in Component 1 it was assumed that immediately at opening time 60 smokers are present and that they each smoke on average two cigarettes per hour. However, the number of smokers present is likely to change with time. For example, when the bar opens no smokers may be present. It could then be assumed that the number of smokers entering the room increases at a constant rate, for example from none at $t = 0$ to 80 at $t = 4$ hours, and remains constant thereafter. Investigation of other relationships between the number of cigarettes smoked and time.

Component 3: Investigation of the effect of variation of other parameters over time. For example, investigation of variation in the flow rate of air into the room (suddenly changed, either increasing or decreasing) and determination of the lowest possible level of operation of the air ventilation system to ensure that eye irritation threshold concentration is never reached.

Starting Point 3: Special treatment

Near the end of the eighteenth century *Thomas Malthus* suggested a simple model for population growth $\frac{dN}{dt} = kN$

where, $k = a - b$, is a positive constant related to the birth rate, and b is a positive constant related to the death rate for a specific population a and N is the population. There are a number of factors that may influence growth of a population, for example the availability of resources. In recent times, scientists have used the strategy of relocating endangered species to a protected environment to allow them to rebuild their population. A problem that can occur in this situation is the rapid over-population of the protected environment, especially where there is a large supply of food and an absence of predators. Scientist use Malthus's model to predict when intervention might be required to prevent over-population under these circumstances.

Component 1: Consideration of specific values for a and b to evaluate the population of the island over times. Exploration of the relationship between population growth or decline for given values of a and b, in particular the values for a and b that led to population decline, stability and growth. Comment on the limitations of the model as a predictor of population.

Component 2: Modification of Malthus's model to account for limited availability of resources and the existence of predators. A more sophisticated mathematical model for population growth takes into account the availability of resources where:

$$\frac{dN}{dt} = \left(k - cN\right)N$$

and k is related to the birth rate and death rate respectively, and c is a positive constant related to the amount of food available on the island.

Solution of the modified differential equation, interpretation of the solution and comparison of the predictive ability of both models.

Component 3: Investigation of growth models where species population growth is not necessarily smooth and continuous (the assumption of smooth and continuous growth can be valid for large populations, but for smaller populations it is more valid to model the growth as a discrete function. A mathematical model of a population growth where the growth is considered discrete can be developed using difference equations. For example, if the natural increase in a population, in one month, is proportional to the number present in the previous month, this can be represented by $N(t + 1) - N(t) \propto N(t)$. Another situation is where the population increase is due solely to migration at a constant rate C. This could be represented by $N(t + 1) - N(t) = C$).

Use of difference equations to describe the growth of a population similar to those considered in Components 1 and 2. Consideration of the differences and similarities between using differential equations and difference equations as models for population growth. Investigation of $P(n + 1) = (1 + a - b)\ P(n)$ or similar as a model for population growth, where a is the birth rate and b is the death rate.

Or alternatively: Investigation of the situation where one population $P(t)$, the *predator* population is dependent on another population $N(t)$, the *prey* population, including consideration of the interaction between *predator* and *prey* in the model for population growth (for example, if the natural rate of growth of the *prey* is $\frac{dN}{dt} = kN$

this will be modified in terms of the number of 'contacts' between the two populations which may be considered as proportional to the product *NP*. Thus,

$$\frac{dN}{dt} = k_1 N - k_2 NP$$

For the *predators*, in the absence of prey the population growth of the predators would be

$$\frac{dP}{dt} = -k_3 P$$

When the relationship of both predator and prey is taken into account this equation becomes

$$\frac{dP}{dt} = -k_3 P + k_4 NP$$

Investigation of the predator-prey model for various values of $k_1\, k_2\, k_3$ and k_4 including consideration of the values of $k_1\, k_2\, k_3$ and k_4 required to have stable, decreasing or increasing populations.

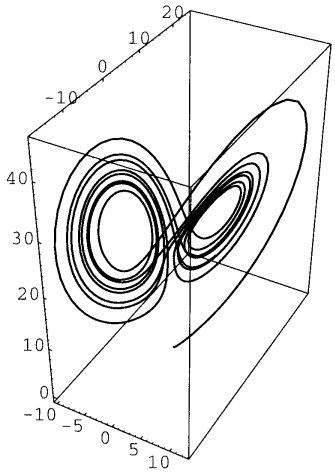
Starting Point 4: Creating chaos with ordinary differential equations

In 1963 *Edward Lorenz*, a meteorologist, discovered a system of non-linear differential equations which could be used to model convection rolls in the upper atmosphere. While the system of equations is relatively simple they can describe complicated behaviour. The chaotic aspect of the system demonstrates that despite being given a set of initial conditions slight variations in these conditions leads to unpredictable outcomes. Lorenz discovered that trajectories of the system, for particular settings, do not settle to a fixed point, do not approach a limit, nor do they diverge to infinity. Lorenz's equations are:

$$\frac{dx}{dt} = \sigma(y - x); \frac{dy}{dt} = \rho x - y - xz \text{ and } \frac{dz}{dt} = xy - \beta z$$

where σ, ρ and β are real constants. The variables *x, y,* and *z* can be considered to be points on a three-dimensional set of cartesian co-ordinates, where $x = x(t)$, $y = y(t)$, and $z = z(t)$ are parametric functions of time.

The following graph shows a parametric plot for these functions for $\sigma = -3$, $\rho = 26.5$ and $\beta = -1$ and initial conditions $x(0) = 0$, $y(0) = 1$ and $z(0) = 0$:



Component 1: Solution of the Lorenz's equations using numerical methods such as Euler's first-order linear approximation method. Use of Euler's method to find approximations for several points using a given sets of values for the parameters σ, ρ and β (the initial values for should be small but non- zero, for example

$$x(0) = y(0) = z(0) = 0.001).$$

Component 2: Lorenz used the values $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$ to model convection currents in the upper atmosphere. He used a computer programme (with less computing power than a graphics calculator) to plot results. Application of appropriate technology to plot sets of points using Euler's method in the form:

$$\frac{dy}{dx} = f(x); y_{n+1} = y_n + hf(x_n) \text{ and } x_{n+1} = x_n + h$$

including consideration of the effect of varying the value of the increment *h* and comment on changes in the resultant graphs. Investigation of the effect of varying the initial value for, and the values of the parameters used to specify the system of differential equations.

Component 3: Investigation of the model dynamical system developed by O. E. Rossler in 1976. The Rossler equations are:

$$\frac{dx}{dt} = -(z + y); \frac{dy}{dt} = x + ay \text{ and } \frac{dz}{dt} = b + xz - cz$$

where the parameters *a, b,* and *c* are real constants.

Investigation of plot projections in any two dimensions, for example *x* and *y*, and consideration of the effect of varying the parameters on these plots (for example, selecting $a = b = 0.2$, then varying the value of *c*. This could start with $c = 3$ and increments of 0.5 until $c = 5$).

Notes

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VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY
41 St Andrews Place East Melbourne Victoria 3002 Australia
TELEPHONE +61 3 9651 4300 FACSIMILE +61 3 9651 4324
EMAIL vcaa@edumail.vic.gov.au
WEB www.vcaa.vic.edu.au