1913 Trigonometry (honours) Question 7

From the equations

$x^2 + y^2 = \frac{x \cos 3\,\theta + y \sin 3\,\theta}{\cos^3 \theta} = \frac{y \cos 3\,\theta - x \sin 3\,\theta}{\sin^3 \theta}$

(i) Eliminate θ and express the result in a simple form.

(ii) Find x and y in terms of θ.

Solution

Rearranging gives the following two equations:

$$\left(x^2 + y^2\right) \cos^3 \theta = x \cos 3\,\theta + y \sin 3\,\theta \qquad (1)$$

$$\left(x^2 + y^2\right) \sin^3 \theta = y \cos 3\,\theta - x \sin 3\,\theta \qquad (2)$$

Squaring and adding the two equations gives a messy looking result, but one than condenses quickly:

$$\left(x^2 + y^2\right)^2 \cos^6 \theta + \left(x^2 + y^2\right)^2 \sin^6 \theta =$$
$$x^2 \cos^2 3\,\theta + 2\,x\,y \cos 3\,\theta \sin 3\,\theta + y^2 \sin^2 3\,\theta + y^2 \cos^2 3\,\theta - 2\,x\,y \cos 3\,\theta \sin 3\,\theta + x^2 \sin^2 3\,\theta$$

$$\left(x^2 + y^2\right)^2 \left(\cos^6 \theta + \sin^6 \theta\right) = x^2 \cos^2 3\,\theta + y^2 \sin^2 3\,\theta + y^2 \cos^2 3\,\theta + x^2 \sin^2 3\,\theta$$

$$\left(x^2 + y^2\right)^2 \left(\cos^6 \theta + \sin^6 \theta\right) = x^2\left(\cos^2 3\,\theta + \sin^2 3\,\theta\right) + y^2\left(\sin^2 \theta + \cos^2 \theta\right)$$

$$\left(x^2 + y^2\right)^2 \left(\cos^6 \theta + \sin^6 \theta\right) = x^2 + y^2$$

$$\cos^6 \theta + \sin^6 \theta = \frac{1}{x^2+y^2} \qquad \mathbf{(3)}$$

The next step is to reduce $\cos^6\theta + \sin^6\theta$ into a function of either sin or cos:

$\cos^6\theta + \sin^6\theta = \left(\cos^2\theta\right)^3 + \left(\sin^2\theta\right)^3$

$= \left(\cos^2\theta + \sin^2\theta\right)\left(\left(\cos^2\theta\right)^2 - \cos^2\theta\sin^2\theta + \left(\sin^2\theta\right)^2\right)$

$\cos^6\theta + \sin^6\theta = \cos^4\theta - \cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta + \cos^2\theta\sin^2\theta$

$\cos^6\theta + \sin^6\theta = \left(\cos^2\theta - \sin^2\theta\right)^2 + \cos^2\theta\sin^2\theta$

$= (\cos 2\theta)^2 + \left(\frac{1}{2}\sin 2\theta\right)^2$

$\cos^6\theta + \sin^6\theta = \cos^2 2\theta + \frac{1}{4}\sin^2 2\theta$

$= \frac{\cos 4\theta+1}{2} + \frac{1}{4}\left(\frac{1-\cos 4\theta}{2}\right)$

$\cos^6\theta + \sin^6\theta = \frac{4\cos 4\theta+4}{8} + \frac{1-\cos 4\theta}{8}$

$\cos^6\theta + \sin^6\theta = \frac{3\cos 4\theta+5}{8}$ **(4)**

Combining (3) and (4) gives the result

$$\frac{1}{x^2+y^2} = \frac{3\cos 4\theta+5}{8} \qquad \textbf{(5)}$$

The next step is to express $x^2 + y^2$ somehow in terms of either x and θ or y and θ.

To do this, go back to equations (1) and (2):

$$\left(x^2+y^2\right)\cos^3\theta = x\cos 3\theta + y\sin 3\theta \qquad (1)$$
$$\left(x^2+y^2\right) = y\cos 3\theta - x\sin 3\theta \qquad (2)$$

$\cos 3\theta \times (1) - \sin 3\theta \times (2):\ \left(x^2+y^2\right)\left(\cos 3\theta\cos^3\theta - \sin 3\theta\sin^3\theta\right) =$

$x\cos^2 3\theta + y\sin 3\theta\cos 3\theta - \left(y\sin 3\theta\cos 3\theta - x\sin^2 3\theta\right)$

$\left(x^2+y^2\right)\left(\cos 3\theta\cos^3\theta - \sin 3\theta\sin^3\theta\right) = x\cos^2 3\theta + x\sin^2 3\theta$

$\left(x^2+y^2\right)\left(\cos 3\theta\cos^3\theta - \sin 3\theta\sin^3\theta\right) = x$

Again, we want to try to remove either the sin or cos function from this expression. Using the identities

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta) \qquad \cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$

$\frac{1}{4}\left(x^2+y^2\right)\left(3\cos 3\,\theta\cos\theta+\cos^2 3\,\theta-3\sin 3\,\theta\sin\theta+\sin^2 3\,\theta\right)=x$

$\left(x^2+y^2\right)(3\cos 3\,\theta\cos\theta-3\sin 3\,\theta\sin\theta+1)=4\,x$

$\left(x^2+y^2\right)(3\cos(3\,\theta+\theta)+1)=4\,x$

$\left(x^2+y^2\right)(3\cos 4\,\theta+1)=4\,x$ **(6)**

Rearranging (5) and (6):

$3\cos 4\,\theta+5=\frac{8}{x^2+y^2}$ (5 a)

$3\cos 4\,\theta+1=\frac{4\,x}{x^2+y^2}$ (6 a)

(5 a) − (6 a) : $\frac{8-4\,x}{x^2+y^2}=4$

Which simplifies to $\frac{2-x}{x^2+y^2}=1$.

This can also be written $x^2+y^2=2-x$. **(7)**

To now express x and y in terms of θ is not too difficult.

$2-x=x^2+y^2 \quad\Rightarrow\quad 3\cos 4\,\theta+5=\frac{8}{2-x}$

$2-x=\frac{8}{3\cos 4\,\theta+5}$

$x=2-\frac{8}{3\cos 4\,\theta+5}$

Which can also be written $x=\frac{6\cos 4\,\theta+2}{3\cos 4\,\theta+5}$

To find y in terms of θ, notice that from (7),
$y^2=-x^2-x+2=-\left(x^2+x-2\right)=-(x+2)(x-1)=(x+2)(1-x)$

$y^2=\left(\frac{6\cos 4\,\theta+2}{3\cos 4\,\theta+5}+2\right)\left(1-\frac{6\cos 4\,\theta+2}{3\cos 4\,\theta+5}\right)$

$y^2=\left(\frac{12\cos 4\,\theta+12}{3\cos 4\,\theta+5}\right)\left(\frac{-3\cos 4\,\theta+3}{3\cos 4\,\theta+5}\right)$

$y^2=\frac{12(\cos 4\,\theta+1)\times 3(-\cos 4\,\theta+1)}{(3\cos 4\,\theta+5)^2}=\frac{36\left(1-\cos^2 4\,\theta\right)}{(3\cos 4\,\theta+5)^2}$

$y^2=\frac{36\sin^2 4\,\theta}{(3\cos 4\,\theta+5)^2}$

$y=\frac{6\sin 4\,\theta}{3\cos 4\,\theta+5}$