

## Suggested Solutions with Marking Scheme to VCAA 2010 Sample Questions

### Question 1

Let  $f: R \rightarrow R$ ,  $f(x) = x^3 + (k+1)x^2 + kx$ .

Solve  $f(x) = 0$  for  $x$ .

#### Solution

##### Method 1

$$x^3 + (k+1)x^2 + kx = 0$$

$$x(x^2 + (k+1)x + k) = 0 \quad 1A$$

$$x(x+k)(x+1) = 0 \quad 1A$$

$$x = 0 \text{ or } x = -k \text{ or } x = -1 \quad 1A$$

##### Method 2

$$x^3 + (k+1)x^2 + kx = 0$$

$$x^3 + kx^2 + x^2 + kx = 0$$

$$x^2(x+k) + x(x+k) = 0 \quad 1A$$

$$x(x+1)(x+k) = 0 \quad 1A$$

$$x = 0 \text{ or } x = -k \text{ or } x = -1 \quad 1A$$

### Question 2

For the simultaneous linear equations:

$$ax + 3y = 0$$

$$2x + (a+1)y = 0$$

where  $a$  is a real constant, find the value(s) of  $a$  for which there are infinitely many solutions.

#### Solution

##### Method 1

$$\begin{vmatrix} a & 3 \\ 2 & a+1 \end{vmatrix} = 0 \quad 1M$$

$$a(a+1) - 6 = 0$$

$$a^2 + a - 6 = 0$$

$$(a+3)(a-2) = 0$$

$$a = -3 \text{ or } a = 2 \quad 2A$$

$y$ -intercept is zero for both lines.

Hence equivalent lines.

For example

When  $a = -3$

$$-3x + 3y = 0$$

$$2x - 2y = 0$$

When  $a = 2$

$$2x + 3y = 0$$

$$2x + 3y = 0$$

##### Method 2

$$m = -\frac{a}{3} = -\frac{2}{a+1}, \text{ y-intercept is zero } 1M$$

$$a(a+1) = 6$$

$$a^2 + a - 6 = 0$$

$$(a+3)(a-2) = 0$$

$$a = -3 \text{ or } a = 2 \quad 2A$$

**Question 3**

Find the equation of the image of  $y = \frac{1}{x}$  under the transformation defined by the

matrix  $\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$  and describe a sequence of transformations that maps the graph of  $y = \frac{1}{x}$

onto the graph of its image.

**Solution****Method 1**

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = 2x \qquad y' = -3y$$

$$x = \frac{x'}{2} \qquad y = -\frac{y'}{3}$$

$$y = \frac{1}{x}$$

$$-\frac{y'}{3} = \frac{1}{\frac{x'}{2}} \qquad 1\text{M}$$

$$\text{Image } y = -\frac{6}{x} \qquad 1\text{A}$$

- Dilation by a factor of 6 from the  $x$ -axis
- Reflection in the  $x$ -axis 1A

**Or** an equivalent sequence of transformations such as

- Reflection in the  $x$ -axis
- Dilation by a factor of 2 from the  $y$ -axis
- Dilation by a factor of 3 from the  $x$ -axis 1A

**Method 2**

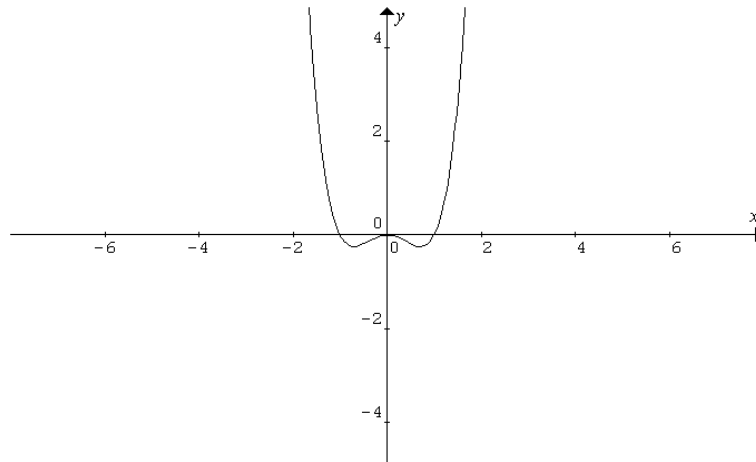
$$y = \frac{1}{x}$$

$$y = -\frac{3}{\left(\frac{x}{2}\right)} \qquad 1\text{M}$$

$$\text{Image } y = -\frac{6}{x} \qquad 1\text{A}$$

**Question 4**

State the subset of  $\mathbb{R}$  for which the graph of the function  $f(x) = x^4 - x^2$  is strictly decreasing.

**Solution**

$$f'(x) = 4x^3 - 2x = 0 \quad 1A$$

$$2x(2x^2 - 1) = 0$$

$$x = 0 \text{ or } x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \text{ or } x = -\frac{\sqrt{2}}{2} \quad 1A$$

$$(-\infty, -\frac{\sqrt{2}}{2}] \cup [0, \frac{\sqrt{2}}{2}] \text{ or equivalent expression. } 1A$$

**Note**

A function  $f$  is said to be strictly decreasing on a given set if for all  $a$  and  $b$  in the set

$a < b$  implies  $f(a) > f(b)$ , hence the use of open and closed endpoints as indicated in the

answer even though  $g'(-\frac{\sqrt{2}}{2}) = g'(0) = g'(\frac{\sqrt{2}}{2}) = 0$ .

### Question 5

Write down a formula that generates all real solutions of the equation  $\sin(x) + \cos(x) = 0$ .

### Solution

$$\sin(x) = -\cos(x)$$

$$\frac{\sin(x)}{\cos(x)} = -1$$

$$\tan(x) = -1 \quad 1A$$

$$x = n\pi - \frac{\pi}{4} \text{ where } n \text{ is an integer.} \quad 1A \quad -\frac{\pi}{4} \quad 1A + n\pi, n \in \mathbb{Z}$$

### Question 6

For the simultaneous linear equations

$$mx + 12y = 12$$

$$3x + my = m$$

find the value(s) of  $m$  for which the equations have

- a** a unique solution
- b** infinitely many solutions.

### Solution

$$\mathbf{a} \begin{vmatrix} m & 12 \\ 3 & m \end{vmatrix} = 0 \quad 1A$$

$$m^2 - 36 = 0$$

$$m \neq 6 \text{ and } m \neq -6$$

$$\text{Alternatively: } m \in \mathbb{R} \setminus \{-6, 6\} \quad 1A$$

$$\mathbf{b} \text{ When } m = 6 \quad \text{When } m = -6$$

$$6x + 12y = 12 \quad -6x + 12y = 12$$

$$3x + 6y = 6 \quad 3x - 6y = -6 \quad 1M$$

Equivalent lines in both cases. This is obvious by looking at  $\frac{12}{m}$ .

$$\mathbf{Answer} \ m = 6 \text{ or } m = -6 \quad 1A$$

### Question 7

Sharelle is the goal shooter for her netball team and during matches has many attempts at scoring a goal. Assume that the success of an attempt to score a goal depends only on the success or otherwise of her previous attempt at scoring a goal.

If an attempt at scoring a goal in a match is successful, then the probability that her next attempt at scoring a goal in the match is successful is 0.84. However, if an attempt at scoring a goal in a match is unsuccessful, then the probability that her next attempt at scoring a goal in the match is successful is 0.64.

In the long term, what percentage of her attempts at scoring a goal are successful?

### Solution

Let  $G$  be scoring a goal and  $G'$  not scoring.

#### Method 1

$$\begin{array}{cc} G_i & G'_i \\ G_{i+1} \begin{bmatrix} 0.84 & 0.64 \\ 0.16 & 0.36 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix} & 1A \end{array}$$

$$0.84x + 0.64(1-x) = x \quad 1M$$

$$-0.8x = -0.64$$

$$x = 80\% \quad 1A$$

#### Method 2

$$\begin{array}{cc} G_i & G'_i \\ G_{i+1} \begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} x \\ 1-x \end{bmatrix} & 1A \end{array}$$

$$x = \frac{b}{a+b} \quad 1M$$

$$= \frac{0.64}{0.8}$$

$$x = 80\% \quad 1A$$

### Question 8

Show that the graph of  $h(x) = \frac{x^n}{e^x}$ , where  $n$  is a positive integer, has a local maximum at  $x = n$ .

### Solution

$$\begin{aligned} h'(x) &= \frac{ne^x x^{n-1} - x^n e^x}{e^{2x}} \\ &= \frac{e^x x^{n-1} (n-x)}{e^{2x}} \quad 1A \end{aligned}$$

For stationary points,  $h'(x) = 0$ .

This implies  $x = n$  1M

$n - x$  determines the sign of the solution.

For  $0 < a < n$ ,  $h'(a) > 0$

and for  $a > n$ ,  $h'(a) < 0$ .

Hence there is a local maximum at  $x = n$ . 1M

**Question 9**

Let  $g: R \rightarrow R$ ,  $g(x) = x^2$ . Show that  $g(u + v) + g(u - v) = 2(g(u) + g(v))$

**Solution**

$$\begin{aligned}
 \text{LHS} &= g(u + v) + g(u - v) \\
 &= (u + v)^2 + (u - v)^2 && 1\text{M} \\
 &= u^2 + 2uv + v^2 + u^2 - 2uv + v^2 \\
 &= 2u^2 + 2v^2 \\
 &= 2(u^2 + v^2) \\
 &= 2(g(u) + g(v)) = \text{RHS as required} && 1\text{M}
 \end{aligned}$$

**Question 10**

For the functions  $f: R \rightarrow R$ ,  $f(x) = e^x + e^{-x}$  and  $g: R \rightarrow R$ ,  $g(x) = e^x - e^{-x}$  show that:

- a**  $[f(x)]^2 = f(2x) + 2$
- b**  $f(x)g(x) = g(2x)$
- c**  $[f(x)]^2 - [g(x)]^2 = 4$

**Solution**

$$\begin{aligned}
 \text{a LHS} &= [f(x)]^2 \\
 &= (e^x + e^{-x})^2 \\
 &= e^{2x} + 2 + e^{-2x} \\
 &= e^{2x} + e^{-2x} + 2 \\
 &= f(2x) + 2 = \text{RHS as required} && 1\text{M} \\
 \text{b LHS} &= f(x)g(x) \\
 &= (e^x + e^{-x})(e^x - e^{-x}) \\
 &= e^{2x} - e^{-2x} \\
 &= g(2x) = \text{RHS as required} && 1\text{M} \\
 \text{c LHS} &= [f(x)]^2 - [g(x)]^2 \\
 &= (e^x + e^{-x})^2 - (e^x - e^{-x})^2 \\
 &= e^{2x} + 2 + e^{-2x} - e^{2x} + 2 + e^{-2x} \\
 &= 4 = \text{RHS as required} && 1\text{M}
 \end{aligned}$$

**Question 11**

Find the average value of  $y = e^x$  over the interval  $[0, 2]$ .

**Solution**

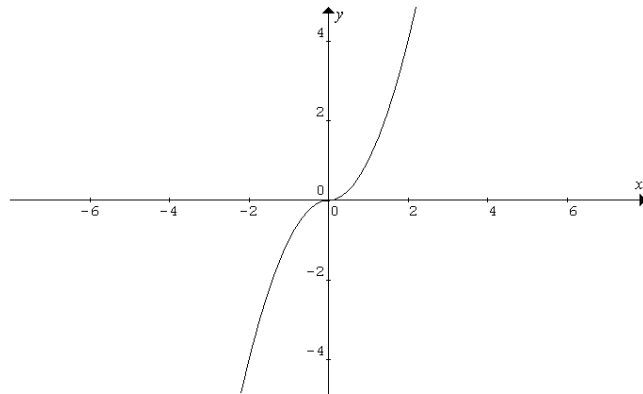
$$\begin{aligned}
 \text{Average value} &= \frac{1}{2-0} \int_0^2 (e^x) dx && 1A \\
 &= \frac{1}{2} [e^x] \\
 &= \frac{1}{2} (e^2 - 1) && 1A
 \end{aligned}$$

**Question 12**

Part of the graph of the hybrid function

$$f(x) = \begin{cases} -x^2, & x \leq 0 \\ x^2, & \text{otherwise} \end{cases}$$

where  $x$  is a real number is shown below:

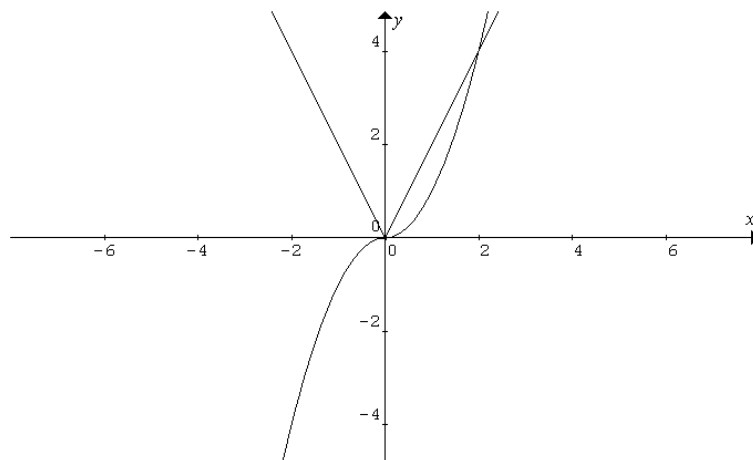


**a** Draw the graph of the derivative function  $f'$  on the same set of axes

**b** Write down a rule for the derivative function.

**Solution**

**a**



Right-hand branch 1A

Left-hand branch 1A

$$\mathbf{b} \quad f'(x) = \begin{cases} -2x, & x \leq 0 \\ 2x, & \text{otherwise} \end{cases}$$

Correct rule and domain                      2 x 1A

alternatively :  $f' : R \rightarrow R, f'(x) = 2x \operatorname{sgn}(x)$ .

Correct domain                      1A

Correct rule                      1A

### Note

The sgn function is defined by:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{where } x > 0 \\ 0 & \text{where } x = 0 \\ -1 & \text{where } x < 0 \end{cases}$$

The hybrid function  $f$  is differentiable at  $x = 0$  with  $f'(0) = 0$ , so  $f'$  can either be specified as a hybrid function, or alternatively using the sgn function, as indicated in the answer, in which case  $f'(x) = 2x \operatorname{sgn}(x)$ .

### Question 13

The speed  $v$ , in metres per second, of an object moving in a straight line is given as a function of time  $t$ , in seconds, by  $v(t) = \frac{24}{t+1}$  where  $t \geq 0$ .

- a** State the initial speed of the object.
- b** Find the values of  $t$  for which the speed is *less than* 2 metres per second.
- c** Find the distance travelled by the object in the first 10 seconds.

### Solutions

$$\mathbf{a} \quad v(0) = 24 \text{ m/s} \quad 1A$$

$$\mathbf{b} \quad \frac{24}{t+1} < 2$$

$$24 < 2t + 2$$

$$t > 11\text{s} \quad 1A$$

$$\mathbf{c} \quad \int_0^{10} \left( \frac{24}{t+1} \right) dt \quad 1M$$

$$= 24 \left[ \log_e (t + 1) \right]_0^t, t > 0$$

$$= 24 \log_e (11) \text{ m} \qquad 1 \text{ A}$$